ULTRASTIFF ELASTIC COMPOSITES VIA NEGATIVE STIFFNESS INCLUSIONS, AND MATERIAL STABILITY IMPLICATIONS

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Summary Composite materials of extremely high stiffness, far higher than that of either phase and exceeding all standard bounds, can be produced by using negative stiffness inclusions. We show this via several exact solutions within linearized and also fully nonlinear elasticity, and via the overall modulus estimate of a variational principle shown to be valid in this case. Negative-stiffness materials are unstable alone, but these composites can be stable under appropriate conditions.

INTRODUCTION

We consider elastic two-phase composite materials comprised of isotropic phases. For such materials, the Voigt bound provides an upper limit on the elastic stiffness of the composite material. If the composite material is macroscopically isotropic, the Hashin-Shtrikman bound provides a tighter (lower) upper bound on the composite elastic stiffness. These bounds are based on the assumption that the elastic materials comprising the composite's phases are both positive-definite. Lakes and Drugan [1] showed that if this assumption is relaxed, composite materials having elastic stiffnesses far higher than these bounds, and than that of either phase, can be produced. For the case of matrix-inclusion composites in which the matrix material is positive-definite, this will occur if the inclusions, which may have a very small volume fraction, have a specifically-tuned negative stiffness. We show this by several exact solutions within linearized and also fully nonlinear elasticity, and also via the overall modulus estimate of a variational principle that we show to be valid in this case of negative stiffness inclusions. Negative-stiffness materials are not stable by themselves, but we present arguments and preliminary results showing that a composite material of the type just described can have overall stability under the appropriate conditions.

EXACT SOLUTIONS AND VARIATIONAL ESTIMATES SHOWING ULTRAHIGH STIFFNESS

We demonstrate that a composite material can have an extremely high stiffness via several exact solutions within linearized and fully nonlinear elasticity. A brief summary of some results is given here; detailed derivations are in [1]. The simplest problem analysed is that of a spherical body containing a concentric spherical inclusion, because of the complete clarity of the results. It is shown in [1] that the overall bulk modulus $\bar{B}$ of the composite sphere is

$$\bar{B} = \frac{B_1(1+2c_1) + 2B_2(1-c_1)(1-2\nu_2)/(1+\nu_2)}{(1-c_1)B_1/B_2 + 2(1-2\nu_2)/(1+\nu_2) + c_1}\ , \tag{1}$$

so that the overall bulk modulus has an extremely high positive value when the (negative) inclusion bulk modulus is just slightly below the critical value

$$B_1 = -\frac{1}{1-c_1}\left[\frac{2(1-2\nu_2)}{1+\nu_2} + c_1\right]B_2. \tag{2}$$

In these results, B is bulk modulus, ν is Poisson's ratio, c is phase volume fraction, and subscripts 1 and 2 refer to the inclusion and matrix phases, respectively. We will also describe the finite elasticity analysis of this problem.
Another exact solution to a more complex microstructure is that of Hashin [2], who derived the composite bulk modulus of a hierarchical microstructure of coated spheres of different sizes in which the outer layer (phase 2) radius and core (phase 1) radius have a specified ratio for all inclusions. The composite is made by adding inclusions of progressively smaller size until the space is filled. The solution remains valid if the core phase has a negative stiffness. This solution turns out to be identical to (1), so that again the overall bulk modulus of this hierarchical composite has an extremely high positive value when the inclusion bulk modulus is just slightly below the critical value (2).
A more novel, general and practically important result showing that an extremely high composite stiffness results from the appropriate choice of negative inclusion stiffness comes from the following demonstration that an existing variational principle is valid in this case. For simplicity, let us consider an infinite linear elastic composite body consisting of firmly-bonded phases that is loaded only by a body force distribution $\boldsymbol{f}(\mathbf{x})$. Following Hashin and Shtrikman, Willis [3] showed that the governing equations of linear elasticity for the stress and infinitesimal strain tensor fields $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, and the displacement vector field $\mathbf{u}$, can be recast as follows: One introduces a *homogeneous* "comparison" body with fourth-rank modulus tensor (independent of position $\mathbf{x}$) $\mathbf{L}_0$ [and having solutions $\boldsymbol{\sigma}_0(\mathbf{x})$, $\boldsymbol{\varepsilon}_0(\mathbf{x})$, to the same applied $\boldsymbol{f}(\mathbf{x})$], so that, in terms of the actual material's position-dependent modulus tensor $\mathbf{L}(\mathbf{x})$,

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}_0\boldsymbol{\varepsilon}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}), \qquad \boldsymbol{\tau}(\mathbf{x}) \equiv [\mathbf{L}(\mathbf{x}) - \mathbf{L}_0]\boldsymbol{\varepsilon}(\mathbf{x}), \tag{3}$$

where the second equation defines the "stress polarization" tensor field $\boldsymbol{\tau}(\mathbf{x})$. The solution to the elasticity field equations can then be shown to involve solution of the following integral equation for $\boldsymbol{\tau}(\mathbf{x})$:

$$\left(\mathbf{L}(\mathbf{x})-\mathbf{L}_0\right)^{-1}\boldsymbol{\tau}(\mathbf{x})+\int \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\tau}(\mathbf{x}')\mathrm{d}\mathbf{x}' \;=\; \boldsymbol{\varepsilon}_0(\mathbf{x}), \tag{4}$$

where $\boldsymbol{\Gamma}_0(\mathbf{x})$ is a fourth-rank tensor field that is two spatial derivatives of the infinite-homogeneous-body (i.e., comparison body) Green's function.

The important fact for present purposes is that Willis [3] proved that self-adjointness of (4) arises solely from the usual index symmetries of the actual elastic modulus tensor $\mathbf{L}(\mathbf{x})$ and the comparison modulus tensor $\mathbf{L}_0$, and that this self-adjointness immediately implies from (4) the Hashin-Shtrikman (stationary) variational principle

$$\delta\left\{\int\left[\boldsymbol{\tau}(\mathbf{x})\left(\mathbf{L}(\mathbf{x})-\mathbf{L}_0\right)^{-1}\boldsymbol{\tau}(\mathbf{x})+\boldsymbol{\tau}(\mathbf{x})\int \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\tau}(\mathbf{x}')\mathrm{d}\mathbf{x}' - 2\boldsymbol{\tau}(\mathbf{x})\boldsymbol{\varepsilon}_0(\mathbf{x})\right]\mathrm{d}\mathbf{x}\right\}=0. \tag{5}$$

Therefore, this variational principle is valid even when one of the phases of the composite has negative stiffnesses, so long as the actual and comparison elastic modulus tensors have the usual index symmetries, and provided that the comparison modulus tensor is chosen such that a Green's function exists for the body.

Willis [4] showed how to apply (5) to random composite materials. Retaining up through two-point statistical information, and assuming statistical uniformity and ergodicity of the composite, he deduced the stochastic form of (5). Making for simplicity the further assumption of isotropic *distribution* of the phases, the stochastic form of (5) gives the following estimate for the effective modulus tensor of the random composite material having n phases each with modulus tensor $\mathbf{L}_r$:

$$\overline{\mathbf{L}}=\left\{\sum_{r=1}^{n} c_r\left[\mathbf{I}+\left(\mathbf{L}_r-\mathbf{L}_0\right)\mathbf{P}\right]^{-1}\right\}^{-1}\sum_{s=1}^{n} c_s\left[\mathbf{I}+\left(\mathbf{L}_s-\mathbf{L}_0\right)\mathbf{P}\right]^{-1}\mathbf{L}_s, \quad \text{where} \quad \mathbf{P}\equiv\frac{1}{4\pi}\int_{|\boldsymbol{\xi}|=1}\tilde{\boldsymbol{\Gamma}}_0(\boldsymbol{\xi})\mathrm{d}S\;. \tag{6}$$

Here, $\mathbf{I}$ is the fourth-rank identity tensor and the constant fourth-rank tensor $\mathbf{P}$ is given in terms of the Fourier transform of $\boldsymbol{\Gamma}_0(\mathbf{x})$.

Result (6) is thus a variational estimate valid for composites having a negative stiffness phase. To be completely explicit, we now consider two-phase composites consisting of an isotropic matrix containing a random distribution of isotropic inclusions (of arbitrary shape). In this case, (6) simplifies to give the following variational estimates for the overall bulk and shear moduli of the composite, where subscripts 1, 2 indicate inclusion and matrix, respectively, and the unsubscripted quantities are the comparison moduli

$$\overline{B}=\frac{4G\left(c_1B_1+c_2B_2\right)+3B_1B_2}{4G+3\left(c_1B_2+c_2B_1\right)}, \qquad \overline{G}=\frac{G(9B+8G)\left(c_1G_1+c_2G_2\right)+6(B+2G)G_1G_2}{G(9B+8G)+6(B+2G)\left(c_1G_2+c_2G_1\right)}. \tag{7}$$

Let us consider specifically a composite consisting of a positive-definite matrix phase containing inclusions of negative stiffness. A permissible and sensible choice for the comparison moduli is that they equal the matrix moduli. Then (7) show that the variational estimates of the bulk and shear moduli can be made to be arbitrarily large and positive by suitable (negative) choice of the inclusion moduli, namely just slightly below the following critical values

$$B_1=-\frac{4G_2+3c_1B_2}{3c_2}, \qquad G_1=-G_2\frac{3\left(3+2c_1\right)B_2+4\left(2+3c_1\right)G_2}{6c_2\left(B_2+2G_2\right)}. \tag{8}$$

STABILITY ISSUES

The above results demonstrate that a composite material having inclusions with appropriately-tuned negative stiffnesses can exhibit composite stiffnesses far greater than those of either component and of existing bounds. For such a composite material to be useful in structural components, it must be stable. As noted above, materials with negative stiffness are unstable by themselves, but negative stiffness inclusions in a positive-definite matrix can yield a composite material that is stable under appropriate conditions. We will review the known results on stability of elastic materials and composites and discuss how it applies in the present context. We will also present the latest results of our theoretical efforts, currently underway, to prove precisely the conditions under which such composite materials are stable.

References

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