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# NUMERICAL HOMOGENIZATION OF A LOCALLY HYPERELASTIC CONSTITUTIVE LAW

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*Summary* We briefly recall the results in periodic homogenization of nonlinear constitutive laws and address in an empirical way the question of the number of periodic cells one has to consider in a particular non convex case of practical interest. We suggest a numerical approach, for which definite conclusions are yet to be obtained, to simulate an inhomogeneous hyperelastic material. Numerical tests are currently under progress.

## THEORETICAL BACKGROUND

Let us consider a material whose mechanical energy is of the form $W_\epsilon = W(\frac{x}{\epsilon}, \xi)$ where $W$ is 1-periodic with respect to its first argument, $\epsilon$ is a small positive parameter describing the size of the microstructure, $x$ denotes the position in $\mathbb{R}^3$ and $\xi$ is the deformation tensor in $\mathcal{M}_3(\mathbb{R})$. The functional $W$ is nonlinear and for almost all $x \in \Omega$ (a finite domain in $\mathbb{R}^3$) has some convexity properties with respect to $\xi$. An approximation of this energy functional for small values of $\epsilon$ is the "limit" in an appropriate sense when $\epsilon \to 0$. The sense one has to give to this limit depends on the convexity properties of $W(x, \cdot)$.

### Strictly convex case

In the case when $W(\cdot, \xi)$ is uniformly continous, coercive, and when for almost all $x \in \Omega$, $W(x, \cdot)$ is strictly convex, one can prove that for any boundary value problem the displacement $u_\epsilon$ associated to the energy functional $W_\epsilon$ converges strongly in $H^1(\Omega)$ towards a displacement $u^*$ associated to a homogenized energy functional $W^*$ defined as follows :

$$W^*(\xi) = \inf_{u \in E(\xi)} \int_\omega W(x, \nabla u(x))dx \tag{1}$$

where $\omega = [0,1]^3$ and $E(\xi) = \{u \in H^1(\omega)^3 | u(x) = \xi \cdot x + v(x)$ with $v$ a $\omega$-periodic function$\}$. Correspondingly one can define a stress tensor $\pi^*(x) = \dfrac{\partial W^*}{\partial \xi}(\nabla u^*(x))$ as the weak limit of $\pi_\epsilon$ when $\epsilon \to 0$ in $L^2(\Omega)^9$.
We emphasize that in this strictly convex case the homogenized energy functional (1) is obtained with a problem set on a single cell $\omega = [0,1]^3$.

### Quasiconvex case

Consider now an energy functional $W(x, \cdot)$ quasiconvex for almost all $x \in \Omega$. One can also define a homogenized energy functional $W^*$, a displacement limit $u^*$ and an associated stress tensor $\pi^*$ satisfying $u^* = \lim_{\epsilon \to 0} u_\epsilon$ in $L^2(\Omega)^3$ and $\pi^* = \lim_{\epsilon \to 0} \pi_\epsilon$ as a Young measure. However $W^*$ is not given by (1) as shown by Müller in [2]. One has to consider many periodic cells, and in fact infinitely many, as:

$$W^*(\xi) = \lim_{n \to \infty} \frac{1}{8n^3} \inf_{u \in E(\xi)} \int_{\omega_n} W(x, \nabla u(x))dx \tag{2}$$

where $\omega_n = [-n, n]^3$.
This formula gives a convergence result that we propose to numerically investigate for a rubber-like constitutive law.

## NUMERICAL STRATEGY SUGGESTED

We consider the problem $\operatorname{div} \pi_\epsilon = 0$ posed in a finite domain $\Omega$ of $\mathbb{R}^3$ with appropriate boundary conditions, where $\pi_\epsilon = \dfrac{\partial W_\epsilon}{\partial \nabla u_\epsilon}$. We solve a coupled problem consisting in a macro problem and a micro problem providing a numerical constitutive law (ie a stress/strain relation) .

### Extended FE$^2$

Our approach aims at treating high nonlinearities and large deformations, and thus uses a Newton type resolution of the macro problem. The micro problem provides the macro problem with two informations: the homogenized stress tensor and a homogenized stiffness matrix. The Newton method for the macro problem reads at step $n+1$

$$\operatorname{div} \pi_M^n + \operatorname{div} \left( \frac{\partial \pi_M^n}{\partial \nabla u_M^n} : (\nabla u_M^{n+1} - \nabla u_M^n) \right) = 0 \text{ in } \Omega \tag{3}$$

with the same boundary conditions as the original problem (to fix the ideas) and where $\pi_M^n$ and $\dfrac{\partial \pi_M^n}{\partial \nabla u_M^n}$ are respectively given at each Gauss point $X$ by the resolution of the following two micro problems.

**Stress tensor**

At step $n$, given a macroscopic deformation $\nabla u_M^n$ at a Gauss point $X$ and given a micro cell $\omega(X)$, the micro problem provides a microscopic stress tensor $\pi_\mu^n$ solution of

$$\text{div } \pi_\mu^n(x) = 0 \text{ in } \omega(X) \tag{4}$$

$$u_\mu^n(x) = \nabla u_M^n(X).x \text{ on } \partial\omega(X) \tag{5}$$

This problem is posed on a domain whose size is of order $\epsilon$. We recall that this problem is nonlinear and its resolution itself requires an iterative Newton type method, for which one can use the analytical constitutive law $\pi_\mu = \dfrac{\partial W_\epsilon(x, \nabla u_\mu)}{\partial \nabla u_\mu}$. We then define the macroscopic stress tensor $\pi_M(X)$ as the mean value of the solution $\pi_\mu^n$ of (4)-(5).

$$\pi_M(X) = \frac{1}{|\omega(X)|} \int_{\omega(X)} \pi_\mu^n(y) dy \tag{6}$$

If the microstructure does not depend on the macroscopic point considered (that is exactly what we assumed in the first section), the micro cell does not depend either on the Gauss point $X$. That definition of the macroscopic tensor is not new (cf. [3]) and is consistant with the theoretical homogenization results we recalled above.

**Stiffness matrix**

Differentiating (6), we have

$$\frac{\partial \pi_M^n}{\partial \nabla u_M^n}(X) = \frac{1}{|\omega(X)|} \int_{\omega(X)} \frac{\partial \pi_\mu^n(y)}{\partial \nabla u_M^n(X)} dy$$

where $\dfrac{\partial \pi_\mu^n(y)}{\partial \nabla u_M^n(X)}$ is the solution of the PDE (4) once differentiated with respect to its boundary condition (5). The new problem we obtain is then a set of 9 linear PDEs with the same linear operator and nine different second members. In addition we can approximate this linear operator by the stiffness matrix used to solve the problem (4)-(5).

## NUMERICAL TESTS PROPOSED

The strategy described above is being implemented. The main issue of practical interest is the number of cells one should consider for the averaging. Quantifying this for classical rubber-like material examples is the aim of the first tests under progress.

Contrary to the case of linear elasticity where an analytical formula for the homogenization of a chessboard composite is available, no such formula is known in nonlinear elasticity. The goal of our second series of tests is to explore the situation for a composite made of two nonlinear materials.

The talk will give the conclusions concerning these tests.

**References**

[1] Braides A., Defranceschi A.: Homogenization of Multiple Integrals. Oxford University Press, 1999.
[2] Müller S.: Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Rat. Anal. Mech.* **99**:189–212, 1987.
[3] Féyel F.: Multiscale $FE^2$ elastoviscoplastic analysis of composite structures. *Comp. Mat. Sci.* **16**:344–354, 1999.