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PENALTY APPROXIMATION OF IMPACT WITH COULOMB FRICTION

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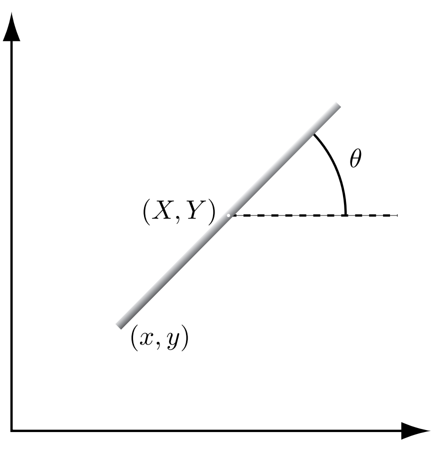
Summary Impact problems with Coulomb friction in a finite number of degrees of freedom are not yet completely understood from the mathematical point of view, and it is useful to possess a range of methods for the constuction of solutions. Moreau, Monteiro-Marques, Stewart used a time-stepping method in order to construct a solution. Here a penalty approximation is proposed; it is simpler than time-stepping, and it allows for more general friction cones and for more transparent mathematical proofs. The non interpenetration constraint is replaced by normal compliance approximation; it is not difficult to obtain estimates on the penalized equation. The passage to the limit on Coulomb's relation as the compliance tends to 0 seems to require the multiplication of Dirac masses by functions which are discontinuous at the coordinate of the Dirac masses. However, precise asymptotics of the approximating problem make it possible to find a relation in the limit between the tangential and normal components of the reaction.

INTRODUCTION

The mathematical understanding of impact with friction is not yet complete, even after the many contributions of Jean-Jacques Moreau, in particular [4] and [3], the work of Manuel Monteiro-Marques [2], [1] and David Stewart's paper of 1998 [6], which was the first to allow for generalized coordinates in this area.
These authors construct a solution through time-stepping; David Stewart needed linear complementarity problems for his analysis and this point of view forced him to approximate three dimensional round friction cones by pyramids.
The purpose of the present paper is to give an alternative construction of solutions, which relies on a standard penalization technique, applied only to the impact part of the model.
The problems considered here use generalized coordinates, and the exact assumptions will be given in the presentation.
Energy estimates are applied in a classical way to obtain uniform bounds on the coordinates and the velocities. The variation of the generalized velocities can be estimated by ideas from convex analysis, though the set of constraints does not have to be convex.
The passage to the limit on the impact complementarity relation is not difficult; the real difficulty comes from the passage to the limit on Coulomb's relation. In this extended abstract, the principle of this passage to the limit will be described in a simple case, in order to make the ideas accessible. But the applicability of the methods goes well beyond the present case.

A SIMPLE MODEL



mass m, length 2ℓ

inertia $J = m\ell^2/3$

gravity g

tangential reaction force $\mathbf{R}_T$

normal reaction force $\mathbf{R}_N$

center of mass (X, Y)

$x = X - \ell\cos\theta, y = Y - \ell\sin\theta$

Figure 1. The mechanical set-up.

If we take into account the impact at only one end of the bar, the mathematical formulation of this problem is as follows,

$$\begin{aligned} m\ddot{X} &= \mathbf{R}_T, \\ m\ddot{Y} &= \mathbf{R}_N - mg, \\ J\ddot{\theta} &= -\ell\big(\mathbf{R}_N\cos\theta - \mathbf{R}_T\sin\theta\big); \end{aligned} \tag{1}$$

the non penetration condition is the complementarity condition

$$y \geq 0, \quad \mathbf{R}_N \geq 0, \quad \langle y, \mathbf{R}_N\rangle = 0, \tag{2}$$

and the Coulomb friction condition is

$$\mathbf{R}_T \in -\mu \operatorname{Sign}(\dot{x})\mathbf{R}_N, \quad \operatorname{Sign}(r) = \begin{cases} \{-1\} & \text{if } r < 0, \\ \{+1\} & \text{if } r > 0, \\ [-1,1] & \text{if } r = 0. \end{cases} \tag{3}$$

HOW TO PENALIZE

The penalization consists in replacing the interpenetration condition by a normal compliance law; if τ is a characteristic time which will tend to 0, we let

$$\mathbf{R}_{N,\tau} = m \max(-y_\tau, 0)/\tau^2. \tag{4}$$

Equations (1) and (3) are not changed and initial conditions are prescribed. The system can now be treated as an integrodifferential monotone problem, where the integral part can be seen as a perturbation of a standard class of equations.

CONVERGENCE

The easy estimates

The first easy estimate is an energy estimate: X_τ, Y_τ, θ_τ and their first derivatives are bounded uniformly in $\tau \in (0,1]$ and in $t \in [0,T]$. It is obtained by multiplying the first equation of (1) by $\dot{X}_\tau$, the second by $\dot{Y}_\tau$ and the third by $\dot{\theta}_\tau$, and by integrating and applying a Gronwall inequality.

The second easy estimate is an estimate on the integral over $[0,T]$ of $|\mathbf{R}_{N,\tau}|$, uniformly in τ as in [5]. This estimate implies another estimate on the integral of $|\mathbf{R}_{T,\tau}|$.

Therefore, one can extract subsequences, and the passage to the limit from (4) to (2) is standard; one should simply remark that the limiting $\mathbf{R}_N$ is a measure.

The difficult estimates

Passing to the limit on (3) is a more delicate problem, since $\mathbf{R}_T$ may have a Dirac mass precisely when $\dot{x}$ vanishes. It is possible to decompose the sequence of approximations $\mathbf{R}_{N,\tau}$ into the sum of two terms: one approximates the continuous part of $\mathbf{R}_N$ and one approximates the atomic part of this measure. On the continuous part, relation (3) passes to the limit. An equation of the form $\mathbf{R}_T^a \in -\mu S \mathbf{R}_N^a$ relates the atomic parts of the limiting measures, where S is a scalar function defined on the support of the atomic part $\mathbf{R}_N^a$ of $\mathbf{R}_N$. The value of S can be calculated from the knowledge of the generalized coordinates and the right and left limits of the velocities at each point supporting a Dirac mass.

CONCLUSION

What good are all these mathematics? First, the proof sketched above can be extended to much more general situations, and the extent of these generalizations will be described in the communication.

Second, the choice of a Coulomb friction model is a decision to simplify reality; how good this decision is can be evaluated by comparison to experiment and also by finding out whether the mathematics make sense or not. Having a range of mathematical methods to solve a problem means that the problem is tractable, and gives arguments in favor of this modelization. It also gives insight into the numerical approximation strategies, and is eventually helpful for understanding the mechanics of the problem.

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