ANISOTROPIC HYPERELASTIC AND PSEUDO-HYPERELASTIC MATERIALS AND APPLICATIONS TO SOFT TISSUE MODELLING

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Summary A consistent mathematical model applicable to pseudo-hyperelastic behaviour of orthotropic soft tissues has been proposed. The model describes loading and unloading processes as well as the dissipation in cyclic process. An appropriate approximation of constitutive relationships in respect to Lagrange strain measure reduces them to the orthotropic model of the Saint-Venant-Kirchhoff. Having in mind available experimental data, our considerations have deliberately been restricted to the plane stress state.

INTRODUCTION

Experimental data clearly show that soft tissues are usually geometrically nonlinear [1]. Those tissues are usually inhomogeneous and anisotropic and only in the case of monotonically loading processes can be described as hyperelastic materials. Indeed, one-dimensional tests show that loading and unloading curve do not coincide, similarly to rubber materials [1,4]. In our earlier papers [3] we underlined that the description of pseudo-elasticity proposed by Fung [1] and employed in many papers is not adequate. In fact Fung's model is not appropriate both from the viewpoint of continuum mechanics and interpretation of experimental data as well as implementation of this model in FEM.

BASIC RELATIONS

We assume the existence of elastic pseudo-potential $P = \overline{P}(\mathbf{C},\mathbf{M},\eta)$, $\overline{P}(\mathbf{C},\mathbf{M},1) = \overline{W}(\mathbf{C},\mathbf{M})$. As usual, by $\mathbf{F}$ we denote the deformation gradient and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green strain tensor, cf. [2]. To describe anisotropy we introduce a parametric tensor $\mathbf{M}$ (or a set of parametric tensors). The function $\overline{W}(\mathbf{C},\mathbf{M})$ is similar to the stored energy function of anisotropic hyperelastic material, cf. [3]. We assume that $J = \det\mathbf{F} > 0$, and $\eta \in (0,1]$ as an additional parameter. In loading processes the material exhibits elastic properties with the energy density $\overline{P}(\mathbf{C},\mathbf{M},\eta)$ where $\eta = const$. Potential $P = \overline{P}(\mathbf{C},\mathbf{M},\eta)$ is called the pseudo-hyperelastic potential and satisfies the following conditions

$$\mathbf{T} = \frac{\partial\overline{P}(\mathbf{C},\mathbf{M},\eta)}{\partial\mathbf{C}} + \frac{\partial\overline{P}(\mathbf{C},\mathbf{M},\eta)}{\partial\mathbf{C}^T}, \quad 0 = \frac{\partial\overline{P}(\mathbf{C},\mathbf{M},\eta)}{\partial\eta} \;, \tag{1}$$

where $\mathbf{T}$ denotes the second, symmetric, Piola-Kirchhoff stress tensor. Conditions (1) ensure the fulfilment of balance of mechanical energy and permit to interpret the unloading process as a behaviour of an elastic material with the elastic energy different from that of the loading process. The parameter η is then an implicit, continuous function of $\mathbf{C}$ and $\mathbf{M}$. We assume that, cf. [4],

$$\overline{P}(\mathbf{C},\mathbf{M},\eta) = \eta\overline{W}(\mathbf{C},\mathbf{M}) + \Phi(\eta) \;. \tag{2}$$

According to the assumption of existence of the natural state and using (1) we get $\overline{W}(\mathbf{I},\mathbf{M}) = 0,\ \Phi(1) = 0,\ \overline{\mathbf{T}}(\mathbf{I},\mathbf{M},\eta) = \mathbf{0}$.

Equations (1) and (2) yield

$$\mathbf{T} = \eta\left[\frac{\partial\overline{W}(\mathbf{C},\mathbf{M})}{\partial\mathbf{C}} + \frac{\partial\overline{W}(\mathbf{C},\mathbf{M})}{\partial\mathbf{C}^T}\right], \quad -\frac{d\Phi(\eta)}{d\eta} = \overline{W}(\mathbf{C},\mathbf{M}) \;. \tag{3}$$

Relation $(3)_2$ defines implicitly the parameter η as a function of $\mathbf{C}$ and $\mathbf{M}$. We assume that the unloading is associated with the decreasing of η, or in other words with the softening of material. The function $\Phi(\eta)$ is assumed to be a concave function; then from Eq. $(3)_2$ we conclude that η is a uniquely defined function dependent on $\overline{W}(\mathbf{C},\mathbf{M})$). In the first, primary, loading cycle we assume that $\eta = 1$ whilst the unloading occurs for a certain value $\mathbf{C}_m$. Then Eq. $(3)_2$ gives: $-\frac{d\Phi(1)}{d\eta} = \overline{W}(\mathbf{C}_m,\mathbf{M}) \equiv W_m$. The last relation means that η depends also on the value of energy W_m. For a complete unloading ($\mathbf{C} = \mathbf{I}$) we assume that η achieves a minimum value η_m. The pseudo-potential assumes then the residual value $\overline{P}(\mathbf{I},\mathbf{M},\eta_m) = \Phi(\eta_m)$. The residual or irreversible value of the energy is interpreted as the energy necessary to provoke the degradation of material. According to the second law of thermodynamics we assume that $\dot{\Phi}(\eta_m) \geq 0$. Here the dot denotes an arbitrary time-like parameter. Equations (3) and (4) imply that it is convenient to postulate a functional dependence for $\Phi(\eta)$ with respect to η, as a function of η and W_m. Let, cf. [4]:

$$-\frac{\mathrm{d}\Phi(\eta)}{\mathrm{d}\eta} = c_1\,\mathrm{Erf}^{-1}\left[c_2(\eta-1)\right] + W_m\,, \tag{4}$$

where c_1 and c_2 are positive constants, and $\mathrm{Erf}^{-1}(.)$ stands for the inverse of the error function Erf(.).

PSEUDO-HYPERELASTIC ORTHOTROPIC MATERIALS IN THE CASE OF PLANE STRESS STATE

According to the general representation of 2D orthotropic scalar function [3], The stored energy function can be written as follows: $\overline{w}(N_i) = \overline{w}\left(\mathrm{tr}(\mathbf{EM}_1), \mathrm{tr}(\mathbf{EM}_2), \mathrm{tr}\mathbf{E}^2\right)$, $\widetilde{W}(J_i) = \widetilde{W}\left(\mathrm{tr}(\mathbf{CM}_1), \mathrm{tr}(\mathbf{CM}_2), \det\mathbf{C}\right)$. Here N_i are the so-called orthotropic invariants of the Lagrange strain measure $\mathbf{E} = (\mathbf{C}-\mathbf{I})/2$ whilst J_i denote the invariants of the right Cauchy-Green tensor. The parametric tensors are defined as follows: $\mathbf{M}_1 = \mathbf{m}_1 \otimes \mathbf{m}_1$, $\mathbf{M}_2 = \mathbf{m}_2 \otimes \mathbf{m}_2$. We assume that each of the stored energy function can be, in the range of relatively small strains measured by **E**, approximated by the stored energy function of the orthotropic SVK (Saint-Venant-Kirchhoff) material

$$\overline{w}(N_i) = w_{SVK}(N_i) + O\left(\|\mathbf{E}\|^3\right), \quad w_{SVK}(N_i) = \frac{1}{2}\left[a_1(N_1)^2 + a_2(N_2)^2 + a_3 N_1 N_2 + a_4 N_3\right]. \tag{5}$$

where a_k $(k = 1,...,4)$ are elastic constants. We propose the following generalization (GSVK) of the stored energy function $(5)_2$:

$$\widetilde{W}(J_i) = \frac{1}{8}\left\{a_1(J_1-1)^2 + a_2(J_2-1)^2 + a_3(J_1-1)(J_2-1) + a_4\left[(J_1+J_2)^2 - 2(J_1+J_2)\right]\right\} + \frac{1}{2}a_5(J_3-1)^2 - \frac{1}{4}a_4\ln(J_3)\,,$$

where a_5 is positive constant. The model GSVK eliminates the fundamental, unrealistic predictions of the model SVK for significant homogeneous deformations in the case of compression. Similarly to Fung's model [1], one can propose the following generalization of $\widetilde{W}(J_i)$ (FGSVK): $\widetilde{W}_F(J_i) = \frac{1}{a}\left(e^{a\widetilde{W}(J_i)} - 1\right)$,where a is a positive constant.

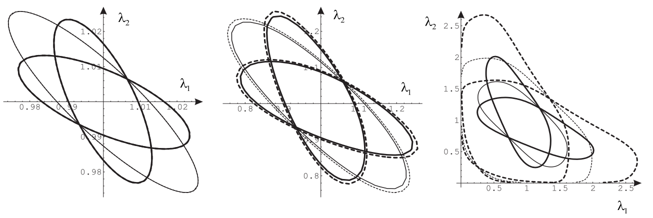

Fig. 1. Comparison of level sets for the stored energy functions of GSVK and FGSVK hyperelastic models. The curves correspond to two-dimensional tests.

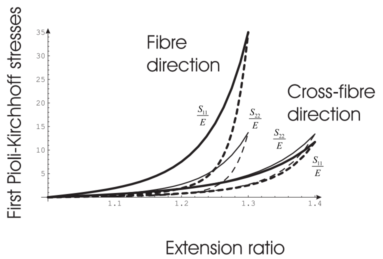

Fig. 2. FGSVK pseudo-hyperelastic model. Comparison of loading and unloading diagrams.

CONCLUSIONS

We have proposed a pseudo-hyperelastic model of soft tissues. The model describes loading and unloading processes as well as the dissipation in cyclic process. An appropriate approximation of constitutive relationships in respect to Lagrange strain measure reduces them to the orthotropic model of the Saint-Venant-Kirchhoff. Then one can correctly interpret the parameters involved in the proposed model and to impose physically motivated restrictions on those parameters.

References

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