SINGULARITIES OF THE FOUR-SIDED ANTIPRISM RING

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Summary An antiprism ring is a simple type of spatial truss structures, which is able to produce finite motions and bifurcation points on their compatibility paths. The structure may, under certain topological and geometric conditions, become both statically and kinematically indeterminate, i.e. an overconstrained mechanism. At singular positions the number of instantaneous degrees-of-freedom can change. The structure is quite sensitive to geometric errors. Depending on the initial geometry and different imperfection parameters, various singularities can occur referring to different cases of mobility of the structure. It may form a rigid structure or produce finite or infinitesimal mechanisms as well.

INTRODUCTION

An antiprism ring is a truss which comprises of two n-sided polygons of vertices. The upper polygon is a closed linkage of bars while the vertices of the lower polygon are foundation nodes. The vertices of the two polygons are interconnected by $2n$ bars in a zigzag layout as illustrated by the four-sided ring in Figure 1a. Generally an antiprism ring is rigid as it satisfies Maxwell's rule for rigidity [1]. However, it can be proved that the existence of a plane of symmetry through a vertex of the upper polygon is a necessary and sufficient condition for mobility if n is even. In this case the truss is both statically and kinematically indeterminate. In this paper the four-sided structure is discussed.

BASIC STRUCTURE

Consider the structure shown in Figure 1a with all bars having a uniform length of 2. The foundation nodes are the corners of a square on the (x, y) plane. In the base position of the structure the upper polygon is also a square. At this position the structure is capable of one finite motion which can be represented by a compatibility path plotted in the space of the variables chosen to describe the geometry of the structure. During the motion the structure can reach a special position shown in Figure 1b where the bars of the upper polygon coincide in pairs. Here the possibility of another motion appears, the compatibility path bifurcates and the structure may change shape to move on a different path. Similar bifurcation phenomena may occur for other mechanisms as well [2, 3, 4].

In our investigation we apply two different sets of variables to describe the bar-assembly. One can assign three Cartesian coordinates for each of the four moving nodes and one constraint condition for each bar setting up the relationship of the variables. Alternatively, one can regard the interconnecting bars rigid and define the position of the moving nodes by one angle each as shown in Figure 1a. Now one constraint condition is formulated for each of the upper bars. A four-dimensional space is required to visualise the compatibility paths. However, as the structure has at least one plane of symmetry during the motion, two of the variables are equal and a three-dimensional plot can be generated. The plot of the paths consists of two intersecting closed loops. The intersection points are bifurcation points.

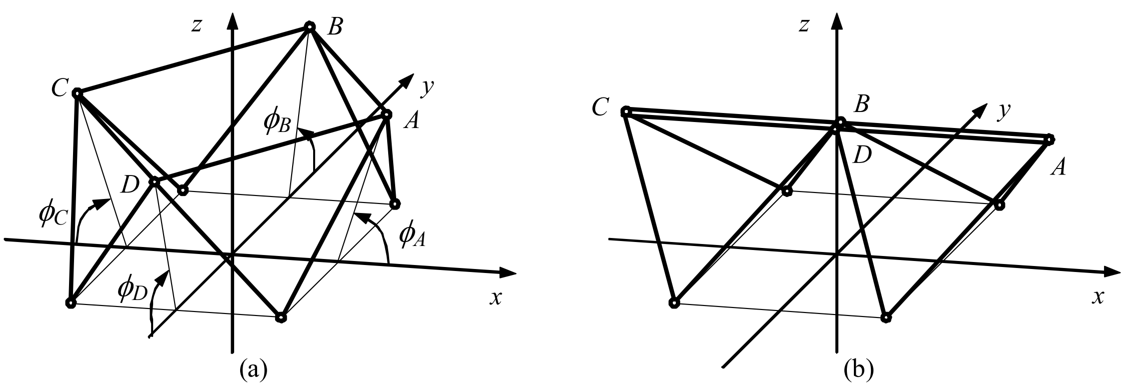

Figure 1: Antiprism ring. (a) Base position. (b) Bifurcation position.

ANALYSIS

In the analysis of the behaviour of the structure we may use either of the variable systems introduced above. Let us regard the structure as a truss and introduce 12 Cartesian coordinates to define the position of the four moving nodes. 12 bars connect the nodes to one another and to the foundations. The equilibrium matrix of the truss can be calculated, which is the basis to determine the static and kinematic determinacy. Now it is a square matrix with 12 rows for the

coordinates of the nodes and 12 columns for the bars. If the matrix is of full rank at a certain position, the structure is rigid. If the row rank decreases by 1 or 2, then 1 or 2 mechanisms become possible. Due to the same number of rows and columns, the column rank will decrease equally corresponding to states of self-stress.
The rank deficiency represents instantaneous degrees-of-freedom. They may correspond to either finite or infinitesimal mechanisms, which cannot be determined by the matrix itself. Finite mechanisms represent motions with no elongations of the bars, while infinitesimal mechanisms cannot be activated without infinitesimal elongations. Compatibility is enforced by applying constraint conditions for the bars, either in terms of Cartesian coordinates or angles. In order to determine whether a mechanism is finite or infinitesimal, it needs to be activated by changing one of the variables. Complying with the all but one of the compatibility conditions, the other variables can be calculated, and the last condition is evaluated. If it is satisfied, the mechanism is finite. If higher than first-order elongations are obtained, the mechanism is infinitesimal.

IMPERFECTIONS

All real structures are subject to geometric imperfections due to manufacturing errors or wear, etc. Such imperfections may have significant effect on the behaviour of the structure, especially if it is an overconstrained mechanism. Imperfections may render a mobile bar-assembly rigid, reduce the finite mobility to an infinitesimal one, or even preserve the finite motion. Imperfections can be defined for all bars as length errors in general. It is beyond the scope of this paper to examine this multi-parameter problem in its entirety, hence here we present a few combinations of imperfections to illustrate different cases of mobility.
Consider first the base configuration shown in Figure 1a without imperfections. The assembly has one finite mechanism and one state of self-stress. The equilibrium matrix has a rank of 11, i.e. a rank deficiency of 1. In the special position shown in Figure 1b the rank decreases to 10 and the compatibility path bifurcates. Now there are two instantaneous degrees-of-freedom, both corresponding to finite mechanisms.
Consider now imperfections applied to the bar-assembly in its base position. An arbitrary set of imperfections will typically make it rigid reducing the compatibility path to a single point. Special combinations can be found, such as a uniform elongation of the upper bars, which will not reduce the mobility.
More cases can be examined in the bifurcation position of the assembly. For brevity, we define a set of imperfections by defining the modified position of the nodes at which the structure is assembled strain-free. Three examples are shown to illustrate different types of mobility. Consider first the case when the four moving nodes are simultaneously shifted by a small extent along axis y keeping the (y, z) plane of symmetry and modifying the supporting bars accordingly. The coincidence of the upper bars is preserved, the rank deficiency remains 2, and both degrees-of-freedom correspond to finite mechanisms, and hence the bifurcation point indeed remains a bifurcation point. However, if the shift is applied in the x direction keeping (x, z) plane of symmetry, one of the finite mechanisms will be reduced to an infinitesimal one. In spite of the singularity of this point, now there is no bifurcation. If shifts are applied in both directions, both mechanisms will become infinitesimal. In this case the compatibility path is a single point though the assembly has no first-order rigidity.

CONCLUSIONS

The examination of the four-sided antiprism ring has shown that the mobility of the assembly is greatly affected by small perturbations. Various types of mobility can be obtained. While the initial structure was statically and kinematically indeterminate, perturbations can render it rigid, i.e. determinate. In the singular position one or both finite mechanisms can be reduced to infinitesimal mechanisms while the assembly is still statically and kinematically indeterminate. That is, the singularity of the system remained while the bifurcation point and compatibility path(s) disappeared, quite contrary to the non-overconstrained linkages, where perturbations typically produce regular paths in the neighbourhood of bifurcation points.

References

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