SPONTANEOUS GENERATION OF INERTIA-GRAVITY WAVES BY BALANCED MOTION

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Summary The spontaneous generation of inertia-gravity waves by balanced motion leads to fundamental limitations in the accuracy of balanced models. In the standard quasi-geostrophic regime (with small Rossby number and order-one Burger number) the amplitude of the inertia-gravity waves can be expected to be exponentially small in the Rossby number. We demonstrate this explicitly by deriving asymptotic estimates for this amplitude in two models described by ordinary differential equations: the five-component model of Lorenz and Krishnamurthy; and a model describing the evolution of sheared disturbances in a three-dimensional Boussinesq fluid. In both cases the asymptotic estimates are confirmed by numerical experiments. The concepts and techniques used in the analysis (optimal truncation of asymptotic series, Borel–Laplace transform, complex-time dynamics) help clarify a number of issues in balanced dynamics and initialization. Their relevance to more realistic models of the atmosphere and oceans is discussed.

INTRODUCTION

The rapid rotation and strong stratification of the atmosphere and oceans at mid-latitudes lead to a clear separation between the advective time scales and the inertia-gravity-wave (IGW) time scales. This time-scale separation, characterized by a small parameter $\epsilon \ll 1$ (essentially the Rossby number), implies that the flow can remain close to a balanced state, free of IGWs. Such a state is best thought of as a manifold of reduced dimensionality in the state space of the system — a slow manifold. Balanced models (e.g. the quasi-geostrophic model) then result from the projection of the primitive equations onto such a manifold, and initialization procedures amount to the projection of initial data.

Using power-series expansions in ϵ, one can in principle obtain a hierarchy of slow manifolds, $\mathcal{M}_n$ say, by truncation at some power ϵ^n. Trajectories of the primitive equations then remain (for a finite time) within an $O(\epsilon^n)$ distance of $\mathcal{M}_n$. Were this procedure to converge, one could define *the* slow manifold $\mathcal{M}_\infty$, an exactly invariant manifold on which the motion is entirely devoid of IGWs. However, it has become clear that such an exactly invariant slow manifold does not exist in general, and that balanced motion, however well initialized, spontaneously generates IGWs. This is consistent with the definition of the (approximately invariant) $\mathcal{M}_n$ for arbitrary n because the expansion procedures are divergent, and because the amplitude of the IGWs that are generated is smaller than any order ϵ^n. Typically, one expects wave amplitudes to be exponentially small, scaling like $\exp(-\alpha/\epsilon)$ for some $\alpha > 0$.

The conclusions just outlined have been drawn using a combination of numerical and analytical results, mostly for low-order models (e.g. [1, 2] and references therein). The analytic results are mainly upper bounds on IGW amplitudes. What appears to be lacking, however, are explicit estimates of these amplitude in the regime most relevant to geophysical fluids, namely the quasi-geostrophic regime, with small Rossby and Froude numbers, both of a similar order of magnitude. This paper reports on the asymptotic derivation of such estimates in two simple models [4, 5]. To capture the amplitude of IGWs spontaneously generated by balanced motion, the techniques of exponential asymptotics, or asymptotics beyond all orders, must be used. By associating the generation of IGW to a Stokes phenomenon, these reveal the importance of considering complex values of the time variable t and, specifically, of identifying the complex values of t for which the balanced motion becomes singular.

LORENZ'S FIVE-COMPONENT MODEL

The system of ordinary differential equations

$$\dot{u} = -vw + \epsilon bvy, \quad \dot{v} = uw - \epsilon buy, \quad \dot{w} = -uv, \quad \epsilon\dot{x} = -y, \quad \epsilon\dot{y} = x + buv, \tag{1}$$

was derived by Lorenz by truncation of the shallow-water equations (see [3]). In the quasi-geostrophic regime, with $\epsilon \ll 1$ and $b = O(1)$, it is easy to derive equations for the slow manifolds $\mathcal{M}_n$; these are given by the (asymptotic but divergent) series

$$x = -buv + \epsilon^2 b(uv^3 - u^3v - 4uvw^2) + \cdots, \quad y = \epsilon b(u^2 - v^2)w + \cdots.$$

Truncating these expansions optimally, one can define the balanced contribution $(x_{\mathsf{bal}}, y_{\mathsf{bal}})$ to (x, y) and, by subtraction, the IGW contribution, expected to be of the form

$$(x_{\mathsf{igw}}, y_{\mathsf{igw}}) \approx C(\cos(t/\epsilon + \phi), \sin(t/\epsilon + \phi))$$

for some C and ϕ.

Now, the evolution of a balanced initial condition, with $C = 0$, leads to the generation of IGWs and thus to an exponentially small $C > 0$. The change in C occurs abruptly, in an $\epsilon^{1/2}$ neighbourhood of the intersection of the real t-axis with a Stokes line joining complex conjugates poles of (u, v, w) in the complex t-plane. Using exponential asymptotics, we show in [4] that the amplitude of the waves generated is

$$C \sim \epsilon^{-2} f(b) \exp(-\alpha/\epsilon),$$

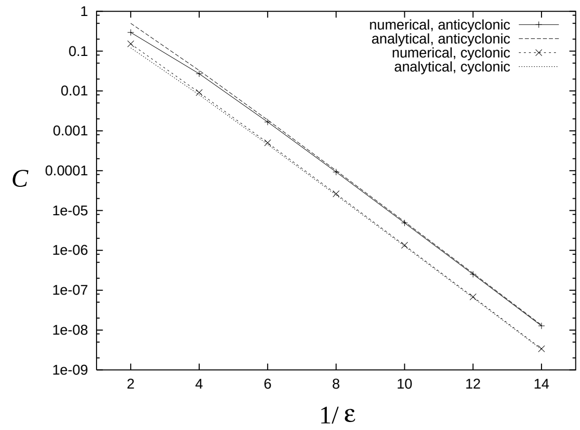

Figure 1. IGW amplitude C as a function of $1/\epsilon$ for sheared disturbances.

where α is the distance of the poles of (u, v, w) to the real t-axis, and the nonlinear function $f(b)$ is determined by a convergent recurrence. Numerical experiments confirm this result.

SHEARED DISTURBANCES

In joint work with I. Yavneh [5], we consider the spontaneous generation of IGWs in exact (nonlinear) solutions of the 3-D Boussinesq equations. Specifically, we examine the evolution of sheared disturbances superimposed to a horizontal Couette flow $\mathbf{u} = (\Sigma y, 0, 0)$. The vertical vorticity of these disturbances has the form

$$\xi(t) \exp[\mathrm{i}(kx + ly + mz)] + \mathrm{c.c.},$$

with $l = -\Sigma kt$ as a result of the shear. The complex amplitude $\xi(t)$ satisfies a linear second-order inhomogeneous equation.
As in Lorenz's model, a balanced contribution to $\xi(t)$ can be defined using an optimally truncated asymptotic series with $\epsilon = |\Sigma|/f$ as small parameter; the remainder takes the form of waves, with

$$\xi_{\mathsf{igw}}(t) \sim C \exp\left(\mathrm{i} \int \omega \, \mathrm{d}t/\epsilon + \phi\right),$$

where ω/ϵ is the IGW frequency. Exponential asymptotics indicates that a (balanced) solution with $C = 0$ for $t < 0$ results in

$$C \sim \epsilon^{-1/2} \beta \exp(-\alpha/\epsilon) > 0$$

for $t > 0$. Here, α and β are given explicitly in terms of elliptic functions of m/k and f/N. A comparison between this estimate and the results of numerical experiments in shown in Figure 1. Interestingly, the wave generation is stronger (by an $O(1)$ factor) for anticyclonic shear than for cyclonic shear.

DISCUSSION

The models studied demonstrate that the spontaneous generation of IGWs by balanced motion is inevitable, although extremely weak in the quasi-geostrophic regime. This confirms the non-existence of an exactly invariant slow manifold and sets a fundamental limit to the accuracy of balanced models. Our more recent work on the subject seeks to extend the results to more realistic models and, ultimately, to the full primitive equations. Also of interest is the definition of a (unique) optimal slow manifold, and the derivation of the corresponding balanced models. Progress on these issues will be reported on.

References

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