XXI ICTAM, 15–21 August 2004, Warsaw, Poland

NON-CONVEX HOMOGENIZATION OF INELASTIC COMPOSITES WITH INTERACTION OF MATERIAL AND STRUCTURAL INSTABILITIES ON DIFFERENT SCALES

Martin Becker, Christian Miehe
University of Stuttgart, Institute of Applied Mechanics (Civil Engineering), 70550 Stuttgart, Germany

Summary In the case of structural instabilities on the micro-scale classical homogenization approaches for composites hit their limitations. Based on an incremental setting of finite inelasticity, a postcritical non-convex homogenization approach is considered for micro-heterogeneous inelastic materials. For these materials stability criteria for structural and material instabilities on the micro- and macro-scale and their connection are presented as well as the size effect of the representative volume element in the nonconvex regime.

Incremental constitutive response of the microstructure and micro-to-macro transition

Micro-instabilities significantly alter the macroscopic material behavior. For example micro-buckling of stiff fibers leads to a considerable softening in the macro-response accompanied by localized discontinuities on the macro-scale. Goal of the presentation is to point out stability criteria on the micro- and macro-scale, as well as their interaction. We discuss their influence on the homogenization procedure and show how instabilities can be incorporated into the micro-to-macro transition. We focus on periodic composites with constituents of standard dissipative materials. For this type of material response we consider an *incremental variational formulation* developed in [3,5,7] where the current stresses at time t_{n+1} are governed by an *incremental stress potential* W which can be constructed by the constitutive minimization problem

$$W(\boldsymbol{F};\boldsymbol{X}) = \inf_{\boldsymbol{\mathcal{I}}\in\mathcal{G}} \int_{t_n}^{t_{n+1}} [\dot{\psi}+\phi]dt \quad \text{with} \quad \boldsymbol{\mathcal{I}}(t_n) = \boldsymbol{\mathcal{I}}_n \tag{1}$$

with respect to a generic vector of internal (history) variables $\boldsymbol{\mathcal{I}} \in \mathcal{G}$, constraint to some manifold $\mathcal{G}$. Here, $\psi(\boldsymbol{F},\boldsymbol{\mathcal{I}};\boldsymbol{X})$ is a *constitutive free energy storage function* and $\phi(\dot{\boldsymbol{\mathcal{I}}},\boldsymbol{\mathcal{I}};\boldsymbol{X})$ a *dissipation function* of the material of the microstructure. With the constitutive functional W at hand we apply a deformation driven homogenization procedure where for given macro-deformation $\bar{\boldsymbol{F}}$ the micro-deformation field $\boldsymbol{\varphi}$ is specified by the *incremental minimization problem of homogenization*

$$\bar{W}(\bar{\boldsymbol{F}}) = \inf_{\boldsymbol{\varphi}\in V_{\#}} \bar{W}(\boldsymbol{\varphi}) \quad \text{with} \quad \bar{W}(\boldsymbol{\varphi}) = \frac{1}{|\mathcal{V}|}\int_{\mathcal{B}} W(\boldsymbol{F};\boldsymbol{X})\,dV, \tag{2}$$

which defines the homogenized quasi-hyperelastic potential $\bar{W}$ as the minimum volume average of the microscopic potential W with respect to the micro-deformation field $\boldsymbol{\varphi}$. Here $|\mathcal{V}|\bar{W}(\boldsymbol{\varphi})$ is the total incremental energy of the microstructure, the set $V_{\#}$ restricts the micro-deformation to periodic boundary conditions on the on the surface $\partial\mathcal{B}$ of the microstructure.

Material instabilities on the macro-scale

The existence of a minimizing solution of the homogenization principle $(2)_1$ is guaranteed if the variational functional $(2)_2$ is *sequentially weakly lower semicontinuous* (s.w.l.s.). This generalizes statements of finite elasticity as outlined in [1,8] to the finite incremental response of generalized standard materials. A loss of *macroscopic rank-1-convexity* implies a lack of *quasiconvexity* and hence a violation of the s.w.l.semincontinuity condition for the homogenized potential. Consequently the existence of minimizing solutions for $(2)_1$ is no longer ensured. An algorithmic control of infinitesimal rank-1-convexity bases on an analysis of the *acoustic tensor* $\{\bar{Q}\}_{ac} = \{\partial^2_{\bar{\boldsymbol{F}}\bar{\boldsymbol{F}}}\bar{W}(\bar{\boldsymbol{F}})\}_{a}{}^{B}{}_{c}{}^{D}\bar{N}_B\bar{N}_D$ as follows

$$\min_{||\bar{\boldsymbol{N}}||=1} \{\det[\bar{\boldsymbol{Q}}]\} \begin{cases} > 0 & \text{for strictly infinitesimal rank-1-convex } \bar{W}(\bar{\boldsymbol{F}}) \\ \leq 0 & \text{for not strictly infinitesimal rank-1-convex } \bar{W}(\bar{\boldsymbol{F}}). \end{cases} \tag{3}$$

For materials being initially homogeneous over all scales a loss of macroscopic rank-1-convexity indicates the development of fine-scale microstructures. In the case of a priori micro-heterogeneous materials a loss of rank-1-convexity on the superior homogeneous scale is due to the development of structural instabilities on the micro-scale.

Structural instabilities on the micro-scale and interaction between micro- and macro-instabilities

We define a criterion for the analysis of structural instabilities within the incremental setting of finite inelasticity. Structural instabilities like the buckling of stiff fibers may occur on the microscale if the *infinitesimal structural stability condition*

$$\bar{W}(\bar{\boldsymbol{F}},\boldsymbol{\varphi}) - \bar{W}(\bar{\boldsymbol{F}},\boldsymbol{\varphi}+\delta\boldsymbol{\varphi}) = \frac{1}{2|\mathcal{V}|}\int_{\mathcal{B}} \nabla\delta\boldsymbol{\varphi} : \partial^2_{\boldsymbol{F}\boldsymbol{F}} W(\bar{\boldsymbol{F}},\boldsymbol{\varphi}) : \nabla\delta\boldsymbol{\varphi}\,dV > 0 \tag{4}$$

is violated. Thus any infinitesimal perturbation $\delta\boldsymbol{\varphi}$ induces an increase of the averaged infinitesimal energy if the micro-deformation $\boldsymbol{\varphi}$ describes an equilibrium state of the microstructure. Condition (4) is satisfied for *positive definite tangent moduli* $\partial^2_{\boldsymbol{F}\boldsymbol{F}}W$ in the whole domain of the microstructure. A macroscopic structural stability criterion can be defined in a similar way. The interaction between micro- and macro-instabilities for nonlinearly elastic materials has been analyzed in [1], a generalization to inelastic materials can be obtained in the incremental setting. Comparison of the infinitesimal structural stability criteria on the micro- and macro-scales shows that a long-wavelength form $\delta\boldsymbol{F} = \bar{\boldsymbol{p}}\otimes\bar{\boldsymbol{N}} + \nabla\delta\boldsymbol{w}$ of the micro-deformation induces a Maxwell form $\delta\bar{\boldsymbol{F}} = \bar{\boldsymbol{p}}\otimes\bar{\boldsymbol{N}}$ of the macro-deformation. Thus a structural instability on the micro-scale can be detected by an analysis for loss of macroscopic rank-1-convexity (for detailed discussions see [4,6]).

Homogenization of nonconvex incremental stress potentials

Structural instabilities on the micro-scale lead to the problem of the necessary homogenization of a non-convex incremental stress potential. We extend results on Γ-convergence obtained by [2] for the elastic regime to the incremental setting of inelasticity based on the potential W. Thus in order to account for structural instabilities within the homogenization procedure the ensemble averaged potential has additionally to be minimized with respect to the size n_c (n_c being the number of characteristic cells) of the microstructure when computing the homogenized incremental stress potential $\bar{W}$

$$\bar{W}(\bar{\boldsymbol{F}}) = \inf_{n_c} \left\{ \inf_{\boldsymbol{\varphi}} \frac{1}{|\mathcal{V}(n_c)|} \int_{\mathcal{B}(n_c)} W(\bar{\boldsymbol{F}}, \boldsymbol{\varphi}) dV \right\} . \tag{5}$$

Hence for example for a cellular microstructure the number of characteristic cells (i.e. the smallest unit capable of representing the periodic structure of the micro-scale) has to be varied. The relevant representative volume element is then characterized by the number of cells n_c which minimizes the incremental energy at the micro-bifurcation point.

Stability analysis of an inelastic fiber-reinforced microstructure

Figure 1 a) shows a schematic picture of a microscopically fiber-reinforced compression member. In order to determine n_c (in this problem n_c is fully determined by the length l of the microstructure), l is varied and the respective bifurcation points at which the structural instabilities occur on the micro-scale are determined from the macroscopic stress-strain relation (Fig. 1b)). Next the values of the total potential $\bar{W}^{tot}$ (sum of the incremental potentials up to the present time step) at the bifurcation points are determined (Fig. 1c)). Corresponding to eq. (5) the relevant microstructure is the one giving the minimum potential at the bifurcation point (Figure 2b), i.e. the microstructure has to be of a length of $l \geq 10$. Comparing Figures 1 b) and 2 c) reveals that for the relevant microstructure the structural instability on the micro-scale (Fig.1b)) occurs at the same compression parameter of $\lambda^{10}_{crit} = 0.016$ as the macroscopic material instability (Fig.2c)).

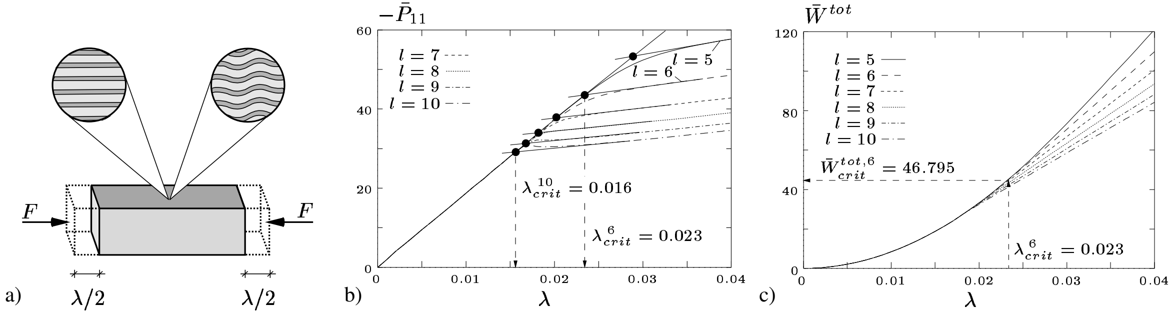

Figure 1. a) Schematic picture of a microscopically fiber-reinforced compression member, subjected to a compressive force F. b) Determination of the bifurcation points; macroscopic 1^{st} Piola-Kirchhoff stress-components $\bar{P}_{11}$ versus compression parameter λ and c) total stress potentials $\bar{W}^{tot}$ versus compression parameter λ for different lengths l of the microstructure.

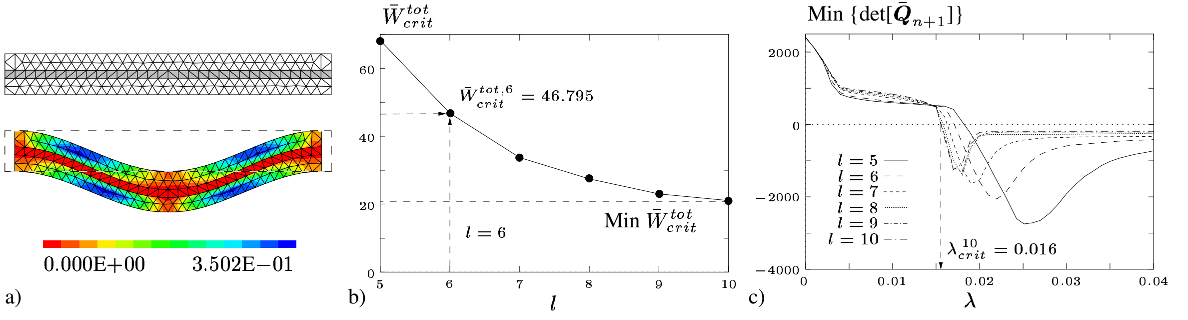

Figure 2. a) Postcritical buckling mode and distribution of equivalent plastic strains at a horizontal compression of 6% for a microstructure of length $l = 8$. b) Determination of relevant size of microstructure; $\bar{W}^{tot}$ at respective bifurcation points versus length l. c) Macroscopic material instability; minimum value of determinant of acoustic tensor versus compression parameter λ.

References

[1] Dacorogna, B.: Direct Methods in the Calculus of Variations. Springer-Verlag, Berlin 1989.

[2] Geymonat, G.; Müller, S.; Triantafyllidis N.: Homogenization of Nonlinearly Elastic Materials, Microscopic Bifurcation and Macroscopic Loss of Rank-One Convexity. *Archive of Rational Mechanics and Analysis* **122**: 231–290, 1993.

[3] Miehe, C.: Strain-Driven Homogenization of Inelastic Microstructures and Composites Based on an Incremental Variational Formulation. *International Journal for Numerical Methods in Engineering* **55**: 1285–1322, 2002.

[4] Miehe, C.; Becker, M.: Homogenization Analysis in Finite Plasticity Based on Incremental Minimization Principles. Material and Structural Instabilities on the Micro- and Macro-Scales and their Interaction. Submitted to *CMAME*, 2003.

[5] Miehe, C.; Schotte, J.; Lambrecht, M.: Homogenization of Inelastic Materials at Finite Strains Based on Incremental Minimization Principles. Application to the Texture Analysis of Polycrystals. *JMPS* **50**: 2123–2167, 2002.

[6] Miehe, C.; Schröder, J.; Becker M.: Computational Homogenization Analysis in Finite Elasticity: Material and Structural Instabilities on the Micro- and Macro-Scales of Periodic Composites and their Interaction. *CMAME* **191**: 4971–5005, 2002.

[7] Ortiz, M.; Stainier, L.: The Variational formulation of viscoplastic constitutive updates. *CMAME* **171**: 419–444, 1999.

[8] Silhavy, M.: The Mechanics and Thermodynamics of Continuous Media. Springer-Verlag, Berlin 1997.