NONLOCAL ESHELBY ENTITIES: A ONE-DIMENSIONAL EXAMPLE

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Summary The following paper describes a nonlocal formulation that differs from the global one in that one out of an infinite possible decompositions has been chosen to express the energy functional in two phases. The first phase indicates what the contribution of each material point (including the boundary) is and the second phase prescribes how all these contributions are to be added up to obtain the total energy of the material body. The formulation is then applied to a number of examples emphasizing the nonlocal effects
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GLOBAL ELASTICITY

Configuration space

The material body $\boldsymbol{B}$ is by definition a one-dimensional manifold with boundary topologically equivalent to a closed interval of $\Re$. Physical space $\boldsymbol{S}$ is identified with $\Re$ itself. A configuration κ is a C^{∞} - embedding of $\boldsymbol{B}$ onto $\Re$:

$$\kappa : B \to \Re . \tag{1}$$

The configuration space Q of the body $\boldsymbol{B}$ is the set of all such embeddings, namely:

$$Q = C^{\infty}\left(B, \Re\right). \tag{2}$$

This set has the natural structure of an infinite-dimensional differentiable manifold.

Global constitutive law

A global elastic constitutive law consists of the assignment of an internal force σ to every configuration κ. In other words, such a constitutive equation is simply a one-form on Q. A material body whose material response is completely characterized by such a one-form is said to be globally elastic. A globally elastic body $\boldsymbol{B}$ is globally hyperelastic if the one-form σ is exact, namely, if there exists a scalar function *W*:

$$W : Q \to \Re \tag{3}$$

such that

$$\sigma = dW . \tag{4}$$

The function *W*, if it exists, will be called the *stored-energy* function of the material body $\boldsymbol{B}$.

NONLOCAL ELASTICITY

An important feature for the representation of global constitutive laws for a material body is obtained by adopting the point of view that although the material may exhibit a completely general global behavior, there is still a physical meaning to be attached to the contribution of each material point to the total energy of the body. In other words, the state of the body, at a material point, is influenced by the state of all points in the body – the so-called *nonlocal effect*. This point of view has been adopted by one of the best established traditions of research in this area, namely that of A.C. Eringen and his collaborators, neatly summarized in a recent book by Eringen [1]. From our perspective, the nonlocal formulation differs from the global one in that one out of an infinite possible decompositions has been chosen to express the energy functional in two phases: the first phase indicates what the contribution of each material point (including the boundary) is, and the second phase describes how all these contributions are to be added up to obtain the total energy of the body.
Let us suppose that we assign to each material point $X \in \boldsymbol{B}$ a scalar function $w(.; X) : Q \to \Re$. For each fixed configuration κ this map can be regarded as a function $w(\kappa;.) \in C^{\infty}(B, \Re)$. In addition, let a particular function $f : C^{\infty}(B, \Re) \to \Re$ be specified. Then the composition

$$W = f \circ w(.\,;.) : Q \to \Re \tag{5}$$

is a legitimate global hyperelastic constitutive law. Note that this construction for the constitutive law is just an example; however, surprisingly enough, every global hyperelastic constitutive law can be expressed in this way, albeit not uniquely. Indeed, if we select a volume element in $\boldsymbol{B}$ (for example, the volume element *dV* induced by the Euclidean ambient of some reference configuration), then for any given hyperelastic constitutive law *W* we can define

$$w(\kappa;X) = \frac{W(\kappa)}{\int_B dV}, \tag{6}$$

and

$$f(\varphi) = \int_B \varphi dV. \tag{7}$$

Then it follows that $f(w(\kappa;X)) = W(\kappa), \forall\, \kappa$. As it can be seen, no generality or specificity has been gained by selecting a particular nonlocal representation except for the fact of a commitment to a particular expression of the constitutive law as a composition of two constitutive factors. This composition, therefore, must be justified in terms of some physical foundation. More specifically, we will apply it later to an example to allow certain forms of laws of material evolution on the grounds that these forms preserve in some way the constitutive identity of the material.
We now proceed to construct an example of the one-dimensional nonlocal Eshelby entities.

Example 1
Eringen's nonlocal elastic solid. Let

$$w(\kappa, X) = \int_B g(X,Y,F(X),F(Y))dV(Y), \tag{8}$$

where g is an ordinary smooth function of its arguments, X and Y are material points, $F(X)$ is the deformation gradient evaluated at the point X, and so on. Let f be selected as per equation (7). Then, according to condition (5) and (6), the following expression for the Piola entity is calculated as follows:

$$<\sigma,\delta\kappa> = <dW,\delta\kappa> = \int_B \left\{ \int_B \left[\frac{\partial g(X,Y,F(X),F(Y))}{\partial F(X)} + \frac{\partial g(Y,X,F(Y),F(X))}{\partial F(X)} \right] dV(Y) \right\} : \delta F(X) dV(X). \tag{9}$$

The expression in curly brackets is interpreted as in [1] as a nonlocal stress tensor Σ.
The corresponding Eshelby entity can be shown to have the following form:

$$<\varepsilon, \delta\beta> = \int_B (F^T\Sigma - GI) : \delta\nabla\beta(X)dV(X), \tag{10}$$

where I is the identity matrix and G is given by:

$$G = \int_B [g(X,Y,F(X),F(Y)) + g(Y,X,F(Y),F(X))]dV(Y). \tag{11}$$

Example 2
Let a reference configuration κ_o consist of an embedding of $\boldsymbol{B}$ onto the closed interval I = [-1,1]. Let $dV = \mu(X)dX$ be a fixed volume element in I, where X denotes the running variable and μ is some positive continuous real function. Let the expression of the constitutive contribution of the point X in the given reference configuration is given by:

$$w(\kappa, X) = \int_{-1}^{1} g(\kappa'(\xi), X)\mu(\xi)d\xi, \tag{12}$$

where g is defined as in the previous example and the prime denotes derivatives with respect to the variable ξ. An example of a possible choice of the of the function g at a point (say, $X = 0$) is:

$$g(\kappa', 0) = \frac{1}{2}E(\kappa' - 1)^2, \tag{13}$$

where E is a material constant. In a different reference configuration, the form $\omega(\kappa, \xi)$ is determined by an appropriate change of configuration represented by some diffeomorphism (for example the differomorphism λ: [a,b] $\rightarrow I$ where $a < b \in \Re$). Note that to complete this example, one still needs to specify the function f and the explicit dependence of ω on X. This point will be deferred to the presentation where the concepts of material uniformity and homogeneity will be discussed.

References

[1] Eringen, A.C., (2002), *Nonlocal continuum theories*, Springer-Verlag.