A DYNAMICAL SYSTEMS ANALYSIS OF THE OVERTURNING OF RIGID BLOCKS

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Summary This work deals with the overturning of a rocking rigid block on an oscillating base, an old fascinating topic which is reconsidered by modern techniques of dynamical systems theory. The paper is divided in two parts: The first is theoretical and concerns the amplitude threshold for contact between stable manifolds and rest position, while the second is numerical and leads to the definition of the "true" safe basin of attraction. These points are somehow correlated, and they go thoroughly into previous authors' insights [2, 3].

INTRODUCTION

The overturning behaviour of rocking rigid blocks has been attracting interest of researchers for a long time, starting with its interest for the estimation of ancient earthquake magnitudes from observations of monuments ruins [4]. Many other practical problems have also been seen to involve this paradigmatic model (see [2, 3] for a brief account), which has a very complex dynamical behaviour in spite of its apparent simplicity. The well-known Housner model [1]:

$$\ddot{\varphi}+\delta\dot{\varphi}-\varphi+\alpha+\gamma\cos(\omega t+\psi)=0,\ \varphi>0,\qquad \ddot{\varphi}+\delta\dot{\varphi}-\varphi-\alpha+\gamma\cos(\omega t+\psi)=0,\ \varphi<0,\qquad \dot{\varphi}(t^+)=r\dot{\varphi}(t^-),\ \varphi=0,\tag{1}$$

is used (δ=damping=0.02, α=block diagonal angle=0.2, γ, ω, ψ=amplitude, frequency and phase of the horizontal excitation). It is based on the assumption that the block can only rock without sliding and uplifting, and undergoes instantaneous impacts (r=restitution coefficient=0.95). It is a quite accurate model for investigating the overturning, which is the practically more interesting outcome. This question has been recently reconsidered by the authors, who studied in detail the heteroclinic bifurcation of the hilltop saddles [2], as well as the question of its optimal control.

Heteroclinic bifurcation is a lower bound for the actual overturning threshold, because below the penetration of the tongues of the overturned attractor into the safe in-well basin is prevented. This is shown in Fig. 1 [3]. Extensive numerical simulations have shown that (Fig. 1): (1) for small γ the block does not overturn (grey) at all; (2) for large γ it directly topples (white) without oscillations. The third intermediate region, where overturning may or may not occur, possibly with a bounded transient, exhibits fractal features.

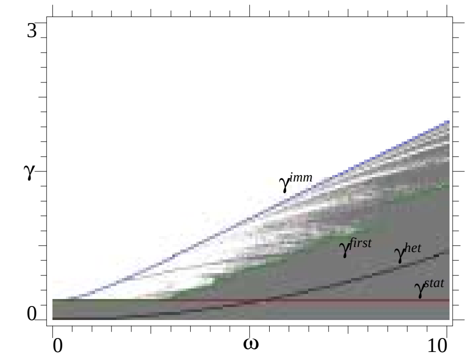

Figure 1.

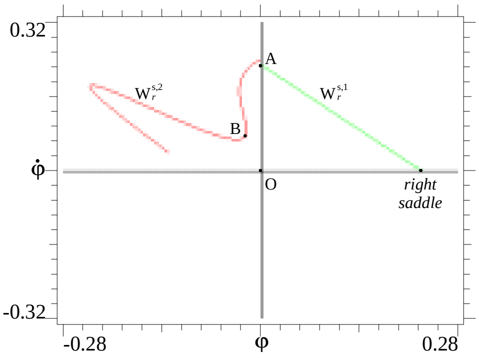

Figure 2. γ=0.7,ω=5 ,ψ=0.

The boundary between high and intermediate regions is the *immediate* overturning threshold γ^{imm}, and corresponds to the minimum γ above which there is overturning without oscillations in the potential well. The boundary between low and intermediate regions is the *first* overturning threshold γ^{first}, and corresponds to the minimum γ above which the rest position topples irrespective of the transient length in the potential well. γ^{imm} has been determined also analytically, while γ^{first} has been analytically approximated from *below* by γ^{het} (heteroclinic bifurcation threshold) and γ^{stat} (static overturning criterion), as shown in Fig. 1. γ^{imm} is also an *upper* bound for γ^{first} (Fig. 1).

The paper [3] has emphasized two important aspects: i) the role of the excitation phase, which is strictly related to the fact that one deals with a single initial condition (the rest position $(\varphi,\dot{\varphi})$=(0,0)) instead of the whole dynamics, and (ii) the role of the invariant manifolds in the immediate overturning. In particular, it has been shown that γ^{imm} corresponds to the first *direct* touching of the stable manifolds W^s with the rest position (i.e., when A touches O for an arbitrary phase, Fig. 2).

THEORETICAL ANALYSIS

Actually, practical interest is rather in determining analytical conditions for *first* overturning, to be possibly pursued through an invariant manifold interpretation of the dynamics occurring for values of $\gamma<\gamma^{imm}$ and finally ending (for a certain phase) with block toppling. It is accomplished in this work by looking for the analytical condition corresponding to touching of point B with the rest position O (Fig. 2). This will allow us to identify a lower upper bound of γ^{first}.

The generic point $(\varphi,\dot{\varphi})$ of the second, left, branch of the perturbed stable manifold $W_r^{s,2}$ of the right saddle (Fig. 2) can be written in the form

$$\varphi=E_1\alpha+\gamma[E_2\cos(\psi)+E_3\sin(\psi)],\qquad \dot{\varphi}=E_4\alpha+\gamma[E_5\cos(\psi)+E_6\sin(\psi)],\tag{2}$$

where the coefficients E_i depends on δ, r, ω and on the time β necessary to this initial condition to impact before asymptotically approaching the right saddle. When the excitation phase ψ varies, the point (2) describes an ellipse in the plane ($\varphi, \dot{\varphi}$) (pseudo phase space), and the point having the maximum value of $\varphi(\psi)$ is given by

$$\varphi_{max}=E_1\alpha+\gamma\sqrt{(E_2^2+E_3^2)}, \qquad \dot{\varphi}_{max}=E_4\alpha+\gamma[E_5E_2+E_6E_3]/\sqrt{(E_2^2+E_3^2)}, \qquad \psi=\mathrm{atan}(E_3/E_2). \qquad (3)$$

The critical condition corresponding to the touching of B with O is mathematically given by $\varphi_{max}=0$ and $\dot{\varphi}_{max}=0$. When α, δ, r, ω are fixed, this is a system of two equations in the two unknowns γ and β. The solutions $\gamma°$ give the amplitude threshold for this critical event. For example, for $\delta=0.02$, $r=0.95$, $\alpha=0.2$, $\omega=5$ the lowest solution is $\gamma°=0.8492$ and $\psi°=0.6476$, and the corresponding manifold-phase-portrait is reported in Fig. 3. Note that $\gamma°$ is below $\gamma^{imm}=1.0297$ (Fig. 4).

The solutions $\gamma°=\gamma°(\omega)$ of the previous system are depicted in Fig. 4. They are constituted by several branches, which are reported with different colors. As expected, the red line represents a better analytical upper bound of γ^{first}.

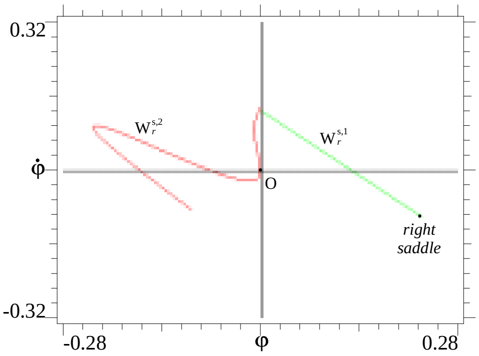

Figure 3. $\omega=5$ $\gamma°=0.8492$, $\psi°=0.6476$.

NUMERICAL INVESTIGATION

In the problem of overturning the initial condition is fixed and the excitation phase is unknown. Thus, safe basins of the attraction in the classical sense (the union of the basins of all in-well attractors), related to a fixed ψ (Fig. 6), do not provide adequate informations. It can, and actually does, occur that for a given phase the block does not overturn, while it topples for a different ψ.

Then, one must look for phase-independent arguments, and the idea is that of defining the "true" safe basin of attraction as the intersection of all classical safe basins when ψ ranges over the period: this is the smallest phase-indendent set of initial conditions which do not entail overturning, and it is therefore reliable from a practical point of view.

To practically determine the "true" safe basin we project the 2D stable manifolds (in the 3D phase space ($\varphi, \dot{\varphi}, t$)) onto the plane ($\varphi, \dot{\varphi}$): the out of projection area surrounding the rest position (0,0) is the "true" safe basin (Fig. 5).

The comparison between Figs. 5 and 6 permits to appreciate the differences between the "true" and the classical safe basins. In particular it is seen how by classical arguments the safety from overturning, here interpreted as the farness from the closest initial condition leading to toppling, is strongly overestimated.

The erosion of the "true" safe basin when the excitation grows is the triggering phenomenon for toppling.

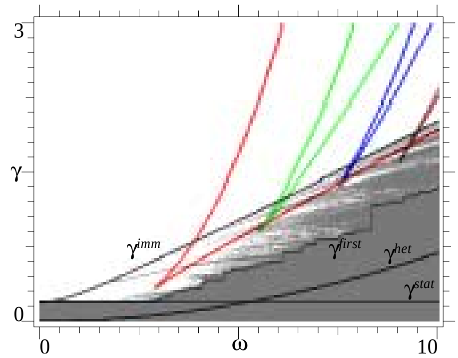

Figure 4.

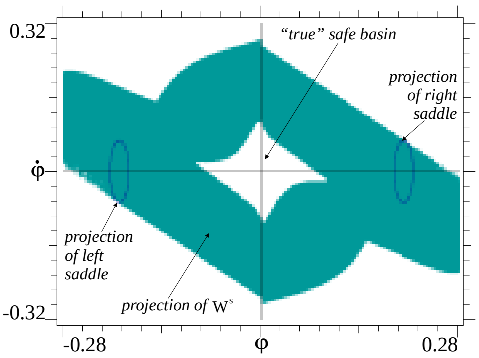

Figure 5. $\omega=5$ $\gamma=0.35$.

CONCLUSIONS

To the authors knowledge, this work is one of the first attempts to make *explicit* the role played by the invariant manifolds on the overturning of rigid blocks, thus providing a theoretical interpretative framework of this important practical phenomenon.

The obtained $\gamma°$ threshold corresponds to touching of the stable manifold with the rest position. The investigation of the possible occurrence of homo/heteroclinic connections between the hilltop and secondary saddles, likely responsible for erosion of the safe basin below γ^{first} and for touching with $\gamma^{first}<\gamma<\gamma°$, is left for future work.

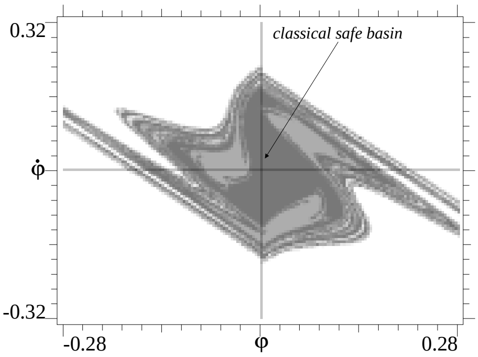

Figure 6. $\omega=5$ $\gamma=0.35$, $\psi=0$.

References

[1] Housner, G.W.: The Behaviour of Inverted Pendulum Structures During Earthquakes, *Bull. Seism. Soc. Am.* **53**: 403-417, 1963.

[2] Lenci S., Rega G.: Heteroclinic Bifurcations and Optimal Control in the Nonlinear Rocking Dynamics of Generic and Slender Rigid Blocks. In press on *Int. J. Bif. Chaos*, 2004.

[3] Lenci S., Rega G.: Overturning Thresholds of a Rocking Block Subjected to Harmonic Excitation: Computer Simulations and Analytical Treatment. *Proc. of "2003 ASME International Mechanical Engineering Congress & Exposition (IMECE'03)"*, Washington, D.C., U.S.A., 16-21 November 2003 (CD-rom). Paper n. IMECE2003-44600.

[4] Milne, J.: Experiments in Observational Seismology, *Trans. Seism. Soc. Japan* **3**: 12-24, 1881.