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A HOMOGENIZATION BASED LAMINATED BEAM THEORY

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Summary The analysis of laminated and sandwich structures is very significant as may be appreciated by the extensive literature devoted to both the development of enhanced beam, plate and shell theories as well as the application of these theories in the design of laminated composite and sandwich structure. There are essentially two aspects of these problems which have driven the theoretical development. First, it is desirable to obtain accurate displacement solutions; second, accurate and complete through-thickness stress/strain distributions are crucial in the prediction of failure/delamination. In both cases a minimal computational effort should be involved. The challenge is that the through-thickness character of the stress, strain and displacement fields are in general non-differentiable and may also be discontinuous; thus modelling the through-thickness response using low-order polynomials will in general lead to poor approximations. The literature dealing with this subject is vast; the reader is referred to [1] which presents a review as well as the complete details of the present work. The de facto standard method of analyzing laminated beams, plates or shells is to use a Timoshenko or Reissner-Mindlin formulation, the so called first-order shear deformation theory, with appropriate shear correction factors and then complete a post-processing calculation in which the in-plane stresses are used in the full equilibrium equations to determine approximations for the transverse shear stresses. The difficulty with this approach is the definition of the shear correction factor and the fact that the post-processing calculation is inconsistent with the beam/plate assumptions used in the analysis. The present work adopts a very different point of view which is completely internally consistent and, as will be shown, leads to an extremely accurate modelling capability.

The present work provides a straight-forward yet extremely accurate approach by which a sequence of beam models is developed; both laminated and sandwich beams are modelled with equal fidelity. This development is based on the assumption that beam response characteristics can be represented in terms of far-field stress and strain solutions corresponding to constant, linear, quadratic, ... , nth degree bending states; such solutions are referred to as Fundamental Solutions, can be determined uniquely and are independent of boundary conditions. Based on the Fundamental Solutions, through-thickness moments of stress and strain are used to obtain definitions of homogenized flexural and shear stiffness, homogenized transverse Poisson's ratio as well as a unique definition of a shear-strain-moment correction. The homogenized flexural stiffness is defined as the ratio of the first moments of σ_x and ϵ_x for a state of pure bending. That is

$$E_x \triangleq \frac{{}_1s_x^1}{{}_1e_x^1} \tag{1}$$

This stiffness does is not reduce to the conventional definition of flexural stiffness. For a single layer material, E_x reduces to Young's modulus. The homogenized transverse shear stiffness is defined as the ratio of the zeroth moments of σ_{xz} and γ_{xz} for a state of linearly varying bending; that is constant transverse shear stress.

$$G_{xz} \triangleq \frac{{}_2s_{xz}^0}{{}_2e_{xz}^0} \tag{2}$$

In the above, the exact shear stress and strain distributions are used to evaluate G_{xz}; therefore no shear correction factor is required. Furthermore, for a single layer material, G_{xz} is identical to the exact transverse shear stiffness, independent of the stress and/or strain distribution. The use of through-thickness stress and strain moments eliminates difficulties commonly associated with discontinuous or non-differentiable solution fields and also means model complexity is independent of the number or type of layers in the beam. A hierarchical sequence of models results from this approach depending on the number of Fundamental States included in a particular model. All models adopt a form similar to that of Classical Timoshenko Beam Theory with the addition of *correction* terms which account for the effects of surface tractions. In addition, the system displacement approximations of the present and the Timoshenko model have different meanings. Finally, a simple, well-defined post-processing step allows determination of complete and precise stress and strain fields based on the Fundamental Solution stress and strain fields used in the model development. The theory is internally self-consistent and the post-processing calculations are based on precisely the same approximations involved in developing the theory

Comparisons between the present approach and a full two-dimensional finite element analysis are completed using the commercial finite element code ANSYS. A sequence of cantilever sandwich-beams subjected to either an end shear load or a uniform lateral pressure is considered; the beam is 100mm long by 10mm thick and the face sheets are 0.5mm thick. The face sheets are assumed to be aluminum; a sequence of core stiffnesses are modelled as degraded aluminum with modulii taking values of 1.0, 0.1, 0.01 times the modulus of aluminum. The ANSYS calculation uses 200x40 bi-quadratic elements. Three examples of typical results are presented in Figures 1, 2 and 3 for the case of a normal surface traction of 0.5 MPa applied on both the upper and lower beam surface. As can be seen there is excellent agreement between the current beam theory and the finite element calculations for the shear stress σ_{xz}, shear strain γ_{xz} and the transverse normal stress σ_z. Complete results show that at the beam mid-length the finite element and beam theory predictions are numerically identical for all stress and strain components. At a distance of 1/2 a beam thickness from the cantilever root the maximum worst case error in the shear stress is approximately 11%; this error be attributed to three-dimensional boundary effects. The through-thickness displacement moments have a maximum worst case error of approximately 1% over the length of the beam .

The present approach provides insights and extensions to existing beam theories including Classical Euler-Bernoulli and Timoshenko beam theory.

References

[1] Hansen, J.S., de Almeida, S.F.M.: A Theory for Laminated Composite Beams. *Final Report* submitted April 2001, FAPES Grant No. 00/06183-0, São Paulo, Brasil.

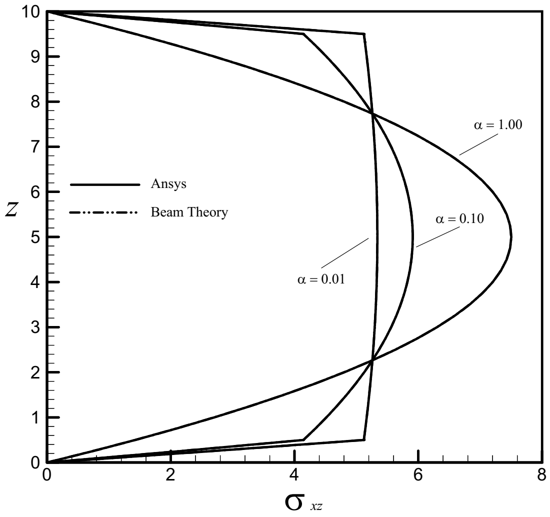

Figure 1. Transverse shear stress σ_{xz} (MPa) through thickness at beam mid-length.

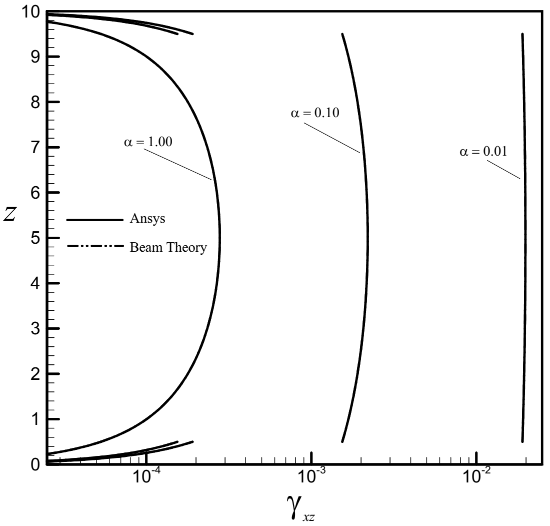

Figure 2. Transverse shear strain γ_{xz} through thickness at beam mid-length.

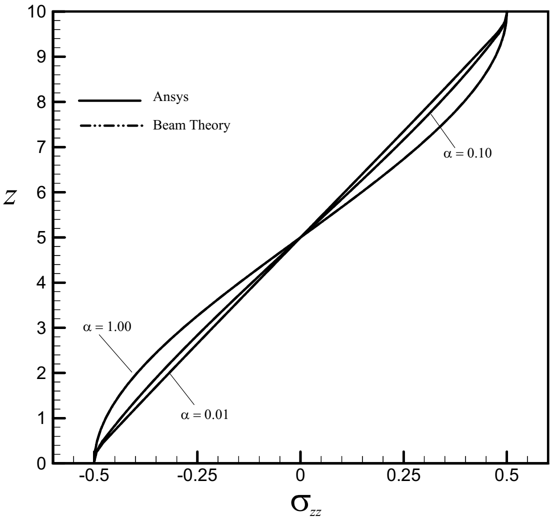

Figure 3. Transverse normal stress σ_{zz} (MPa) through thickness at beam mid-length.