A NEW CONVECTIVE INSTABILITY WITH GROWTH NORMAL TO A BOUNDARY LAYER

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Summary A localized disturbance to an unstable shear layer might grow in the frame of reference of the source of the disturbance (absolute instability), or in frames moving upstream or downstream of the source (convective instability). However, we have discovered a new type of convective instability in which the disturbance grows as it propagates normal to, i.e. out of, a boundary layer.

INTRODUCTION

If an impulsive disturbance is introduced to an unstable shear layer in which the basic flow is predominantly in one direction (the downstream direction), then the disturbance is likely to grow only as it propagates downstream. As the disturbance propagates away, the region surrounding the source is eventually left undisturbed. Nonlinear behaviour will occur some distance downstream depending on the initial amplitude of the disturbance and the spatial growth rates of the shear layer. This behaviour is called convective instability. Alternatively, if the shear layer contains appreciable basic flow in both downstream and upstream directions then a localized disturbance can grow in the frame of reference of the source, leading eventually to nonlinear behaviour at the position of the source. This is called absolute instability.

This distinction between absolute and convective instability was first recognized by Twiss in 1951 in the field of plasma physics. Absolutely unstable flows act like self-excited oscillators, while convectively unstable flows act as spatial amplifiers, and so the nonlinear dynamics of each can be very different. A method for systematically determining which type of instability is present was developed in the 1960s by Briggs and Bers, and has now been used in many fluid flows.

However, in the present work, we have identified a type of convective instability that is fundamentally different to that described above, though it was discovered using the Briggs-Bers method. It is an instability in the direction normal to the shear layer, with disturbances growing exponentially with distance out into the freestream. This is counter-intuitive behaviour because it means that the growth takes place in a region of zero shear, and therefore where the Reynolds stresses are zero. Energy is transferred from the shear layer to the freestream via exponentially divergent eigenfunctions. Again, this is curious, since these 'eigenfunctions' don't satisfy homogeneous boundary conditions.

METHODOLOGY

The linearized stability equations for parallel shear layers can be most conveniently solved in frequency-wavenumber space, and the physical solution reconstructed from inverse Fourier transforms of the form

$$\hat{v}(x,y,t) = \frac{1}{4\pi^2}\int_A\int_F \frac{f(\omega)v(\alpha,y,\omega)}{\Delta(\alpha,\omega)}\exp \mathrm{i}(\alpha x-\omega t)\,\mathrm{d}\omega\,\mathrm{d}\alpha \tag{1}$$

where $\hat{v}$ is a disturbance velocity, v is its Fourier transform, f is the transform of the time-dependent forcing ($f=1$ for the impulsive forcing considered here) and $\Delta=0$ is the dispersion relation for waves satisfying homogeneous boundary conditions. The shear flow $U(y)$ is in the x-direction, the shear is confined to a layer of finite thickness in y and U rapidly approaches a constant value as $y\to\infty$. The inversion contour F must be placed above any singularities in the complex ω-plane to ensure causality, i.e. $\hat{v}=0$ for $t<0$, and A can be placed on the real axis of the complex α-plane.

The Briggs-Bers method allows the propagation properties in physical space to be determined by only considering eigenvalues satisfying $\Delta=0$, without resorting to the evaluation of the integrals. The roots of α satisfying $\Delta=0$ are found for a particular value of ω on F. The locii of these values of α as ω moves along F are plotted in the complex α-plane and are called spatial branches. The behaviour of the spatial branches is investigated as F is lowered towards the real axis in the ω-plane. Spatial branches originating above or below the real α axis correspond to downstream or upstream propagating waves respectively. If a spatial branch from the upper half-plane crosses the real α axis as F is lowered, then there is spatial growth in the downstream direction. If this downstream branch coalesces with an upstream branch from the lower half-plane for a value of ω with $\mathrm{Im}(\omega)>0$, then the flow is absolutely unstable.

THE ROTATING DISC BOUNDARY LAYER

We have investigated the boundary layer flow produced when an infinite disc rotates at constant angular velocity in otherwise stationary fluid. Fluid close to the disc is accelerated in the circumferential direction by viscous stresses. This fluid then spirals outwards due to centrifugal effects, and is replaced by fluid drawn down the axis of rotation. This basic flow is described by von Kármán's similarity solution, and small disturbances are added to this basic flow. The disturbance equations are linearized, a WKB formulation is adopted to account for the weak radial dependence of the basic flow far from the axis of rotation and inverse powers of the Reynolds number are neglected, leading to the Rayleigh equation. Far from the surface of the disc the basic flow is constant and the Rayleigh equation reduces to the Laplace equation with solution

$$w(\alpha,\beta,z,\omega) = C_1 \exp\left(-\sqrt{\gamma^2}z\right) + C_2 \exp\left(\sqrt{\gamma^2}z\right) \quad (2)$$

where C_1 and C_2 are constants of integration, $\gamma^2 = \alpha^2 + \beta^2$ is the square of the total wavenumber, α is the radial wavenumber, β the azimuthal wavenumber, w is the axial velocity component of the disturbance and z is the axial coordinate normal to the disc. The symbol $\sqrt{\ }$ denotes the square-root with positive real part. Therefore, the homogeneous boundary condition $\lim_{z\to\infty} w = 0$ is obtained by taking $C_1 \neq 0$ and $C_2 = 0$. We consider the solution for given real values of β; our definition of the square-root then implies the existence of branch-cuts along the imaginary axes of the complex α-plane (though the imaginary axis is analytic for $-\beta < \mathrm{Im}(\alpha) < \beta$).

Lingwood [1] has shown that this flow is absolutely unstable for $0 < \beta < 0.2652$. The present study was motivated by the results of our recent, unpublished, longwave theory for this absolute instability, which show that the pinch-point becomes asymptotically close to the imaginary axis of the α-plane as $\beta \to 0$, and that spatial branches cross the imaginary axis. The part of the spatial branch on the 'wrong side' of the imaginary axis can only be accessed by moving the branch-cut due to the square-root away from the imaginary axis, or, equivalently, taking $C_1 = 0$ and $C_2 \neq 0$. We argue that these apparently unphysical solutions correspond to a new convective instability with growth in the wall-normal direction.

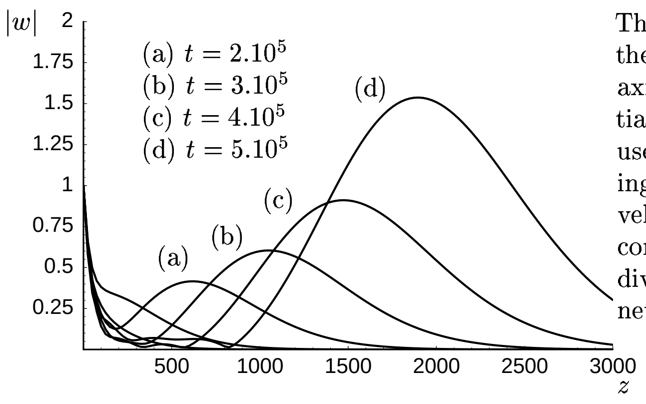
The classic saddle-point analysis for estimating the behaviour of (1) at large times in different reference frames is modified by including the exponential terms from (2) in the phase function:

$$\int_A \int_F \frac{w(\alpha,\beta,z,\omega)}{\Delta(\alpha,\beta,\omega)} \exp \mathrm{i}(\alpha r + \beta\theta - \omega t)\, \mathrm{d}\omega\, \mathrm{d}\alpha = \int_A -\frac{2\pi \mathrm{i}}{\Delta_\omega} \exp\left[\mathrm{i}(\alpha r + \beta\theta - \omega t) - \sqrt{\gamma^2}z\right]\, \mathrm{d}\alpha \quad (3)$$

where the ω integral has been evaluated using the residue theorem and $\omega = \omega(\alpha)$ satisfes the dispersion relation. The saddle point of the exponent is now found from

$$\frac{\mathrm{d}\omega}{\mathrm{d}\alpha} = \frac{r}{t} + \frac{\mathrm{i}\alpha}{\sqrt{\gamma^2}}\frac{z}{t}. \quad (4)$$

We consider frames of reference at fixed radius r, that move away from the disc, i.e. $r/t = 0$ and $z/t \geq 0$. For example, at $\beta = 0.007$ the saddle itself crosses the imaginary axis of the α-plane for a finite range of z/t and the maximum growth rate occurs for $z/t = 0.0042$. But can these saddle-point calculations based on exponentially divergent terms in (2) really be correct?



The figure shows the numerical evaluation of (3) where the path A remains the 'correct side' of the imaginary axis, so that only solutions with decaying exponentials satisfying homogeneous boundary conditions are used during the calculation. The envelope of a growing wavepacket propagating into the freestream at a velocity close to $z/t = 0.0042$ can be clearly seen, confirming that in this problem the exponentially diverging eigenfunctions do indeed correspond to a new type of convective instability.

[1] Lingwood R. J.: Absolute instability of the boundary layer on a rotating disk. *J. Fluid Mech.* **299**; 17–33, 1995.