INERTIAL SIMILARITY OF VELOCITY DISTRIBUTIONS IN HOMOGENEOUS ISOTROPIC TURBULENCE

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Summary The one- and two-point velocity distributions of homogeneous isotropic turbulence are obtained as the solutions of the closed set of equations, which are derived from the Lundgren-Monin (1967) equations using the the cross-independence hypothesis. This problem was first reported by Tatsumi (2000) in ICTAM 2000, and new concrete results and strengthend physical arguments are presented in the present version.

CROSS-INDEPENDENCE HYPOTHESIS

Cross-Independence

The cross-independence of two velocities $\mathbf{u}_1 = \mathbf{u}(\mathbf{x}_1, t)$ and $\mathbf{u}_2 = \mathbf{u}(\mathbf{x}_2, t)$ is defined as the independence of their sum $\mathbf{u}_+ = (\mathbf{u}_1 + \mathbf{u}_2)/2$ and difference $\mathbf{u}_- = (\mathbf{u}_2 - \mathbf{u}_1)/2$. Unlike the ordinary independence of $\mathbf{u}_1$ and $\mathbf{u}_2$ which is only valid for large values of the distance $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$, the *cross-independence* is shown to be valid for both large and small values of the distance $\mathbf{r}$. This may be understood from a simple example: $\langle XY \rangle = \langle ((X+Y)/2)^2 \rangle - \langle ((X-Y)/2)^2 \rangle$.

Independence of Small and Large Eddies

It is interesting to note that the cross-independence is essentially identical with Kolmogorov's (1941) basic premise that small-scale eddies characterized by the velocity difference $2\mathbf{u}_-$ are independent from large-scale eddies characterized by $\mathbf{u}_1$ and $\mathbf{u}_2$. Experimental and numerical supports have been given by Sreenivasan et al. (1998) showing that the cross independence is accurately satisfied for the distance $\mathbf{r}$ of order of the inertial range.

ONE-POINT VELOCITY DISTRIBUTION

Cross-Velocity Distributions

For homogeneous turbulence, we define the one- and two-point velocity distributions of the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ by $f(\mathbf{u}_1, t)$ and $f^{(2)}(\mathbf{u}_1, \mathbf{u}_2; \mathbf{r}, t)$ respectively, likewise for the velocity $\mathbf{u}_+$ by $g_+(\mathbf{u}_+, \mathbf{r}, t)$, for $\mathbf{u}_-$ by $g_-(\mathbf{u}_-, \mathbf{r}, t)$, and for the pair of $\mathbf{u}_+$ and $\mathbf{u}_-$ by $g^{(2)}(\mathbf{u}_+, \mathbf{u}_-; \mathbf{r}, t)$. Then, the cross-independence of $\mathbf{u}_1$ and $\mathbf{u}_2$ gives the relationship,

$$f^{(2)}(\mathbf{u}_1, \mathbf{u}_2; \mathbf{r}, t)\, d\mathbf{u}_1 d\mathbf{u}_2 = g^{(2)}(\mathbf{u}_+, \mathbf{u}_-; \mathbf{r}, t)\, d\mathbf{u}_+ d\mathbf{u}_- \quad (1)$$

$$g^{(2)}(\mathbf{u}_+, \mathbf{u}_-; \mathbf{r}, t) = g_+(\mathbf{u}_+, \mathbf{r}, t)\, g_-(\mathbf{u}_-, \mathbf{r}, t) \quad (2)$$

One-Point Velocity Distribution

On substitution of the cross-independence relations (1) and (2), the Lundgren-Monin equation for the one-point velocity distribution $f(\mathbf{u}_1, t)$ is simplified as

$$[\partial/\partial t + \alpha(t) \,|\, \partial/\partial \mathbf{u}_1 |^2]\, f(\mathbf{u}_1, t) = 0 \quad (3)$$

$$\alpha(t) = (2/3)\, \nu \lim_{|\mathbf{r}| \to 0} |\partial/\partial \mathbf{r}|^2 \int |\mathbf{u}_-|^2 g_-(\mathbf{u}_-, \mathbf{r}, t)\, d\mathbf{u}_- \quad (4)$$

where $\alpha(t)$ is the inverse diffusion constant. It can be shown further that $\alpha(t)$ is related with the mean energy dissipation rate $\overline{\varepsilon}(t)$ and the energy of turbulence $\overline{E}(t)$ as

$$\alpha(t) = (1/3)\, \overline{\varepsilon}(t) \quad (5)$$

$$\overline{\varepsilon}(t) = \langle \varepsilon(\mathbf{x}, t) \rangle = \nu \sum_{i,j=1}^{3} \langle (\partial u_i(\mathbf{x}, t)/\partial x_j)^2 \rangle = -\, d\overline{E}(t)/dt. \quad (6)$$

$$\overline{E}(t) = \langle E(\mathbf{x}, t) \rangle = (1/2) \langle |\mathbf{u}(\mathbf{x}, t)|^2 \rangle \quad (7)$$

Eq. (3) permits a self-similar solution for the one-point velocity distribution.

$$f(\mathbf{u}_1, t) = f_0(\mathbf{u}_1, t) = (t/4\pi\alpha_0)^{3/2} \exp[-|\mathbf{u}_1|^2 t/4\alpha_0] \quad (8)$$

$$\alpha(t) = \alpha_0 t^{-2}, \quad \overline{\varepsilon}(t) = \varepsilon_0 t^{-2}, \quad \overline{E}(t) = E_0 t^{-1}, \quad \alpha_0 = (1/3)\, \varepsilon_0 = (1/3) E_0 \quad (9)$$

which represents the three-dimensional *normal distribution.*

Inertial Similarity

The equation (3) and the solution (8) of the one-point velocity distribution clearly indicate that it depends upon only one parameter $\alpha(t) = (1/3)\, \overline{\varepsilon}(t)$ and not on the viscosity ν explicitly. This means that the one-point velocity distribution of homogeneous isotropic turbulence obeys the *inertial similarity* of Kolmogorov's sense.

The *inertial normality* of the velocity distribution was first pointed out by Hopf (1952) as a particular solution representing the velocity distribution functional but does not seem to have drawn attention of later researchers. It will be shown below that the two-point velocity distributions also obey the normal distribution, so that the *inertial normality* seems to be a universal character of the statistics of homogeneous turbulence.

It may be interesting to note that the energy dissipation $\alpha(t)$ is expressed in terms of the integral of turbulent fluctuation, so that it actually satisfies the *fluctuation-dissipation* relationship of statistical physics.

Viscous Similarity

It should be noted that the inertial similarity is not a unique consequence of eq. (4) since it permits another limit,

$$\alpha(t)=\nu\gamma(t),\quad \gamma(t)=(2/3)\lim_{|r|\to 0}|\partial/\partial \mathbf{r}|^2\int|\mathbf{u}_-|^2 g_-(\mathbf{u}_-,\mathbf{r},t)\,d\mathbf{u}_- \qquad (10)$$

with finite $\gamma(t)$. This similarity associated with the finite energy dissipation $\varepsilon(t)$ and the viscosity ν constitutes the full-set of Kolmogorov's local equilibrium and may be called the *viscous similarity*.

The *viscous normal distribution* (8) associated with the energy dissipation (10) proportional to the viscosity ν is already familiar for us as the normality of weak turbulence composed of large number of independent small eddies.

TWO-POINT VELOCITY DISTRIBUTIONS

Velocity-Sum Distribution

On substitution of the cross-independence relations into the Lundgren-Monin equation for the two-point velocity distribution, we obtain a closed equation for $f^{(2)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{r},t)=2^{-3}g^{(2)}(\mathbf{u}_+,\mathbf{u}_-;\mathbf{r},t)$. Then, on substitution of (2) and integration with respect to $\mathbf{u}_-$, this equation is reduced to the following equation for the velocity-sum distribution:

$$[\partial/\partial t+(1/2)\alpha(t)|\partial/\partial \mathbf{u}_+|^2]\,g_+(\mathbf{u}_+,\mathbf{r},t)=0 \qquad (11)$$

This equation is identical to eq. (3) for the one-point velocity distribution except for the factor 1/2 of $\alpha(t)$. Thus, its solution is immediately given from eq. (8) as

$$g_+(\mathbf{u}_+,\mathbf{r},t)=g_0(\mathbf{u}_+,t)=(t/2\pi\alpha_0)^{3/2}\exp[-|\mathbf{u}_+|^2 t/2\alpha_0] \qquad (12)$$

Comparison of this result with eq. (8) clearly shows that the velocity-sum distribution (12) is given by the convolution of two independent velocity distributions (8) at the points $\mathbf{x}_1$ and $\mathbf{x}_2$.

Although the *inertial normal distribution* (12) is valid for all values of $\mathbf{r}>0$, it must coincide with (8) in the limit of $\mathbf{r}\to 0$ since $g_+(\mathbf{u}_+,\mathbf{r},t)\to f(\mathbf{u}_1,t)$ in this limit. This implies an abrupt change of $g_+(\mathbf{u}_+,\mathbf{r},t)$ at $\mathbf{r}=0$, but such a discontinuity is replaced by the continuous change under the viscous similarity mentioned above.

Lateral Velocity-Difference Distribution

Integration of the closed equation for $f^{(2)}(\mathbf{u}_1,\mathbf{u}_2;\mathbf{r},t)=2^{-3}g^{(2)}(\mathbf{u}_+,\mathbf{u}_-;\mathbf{r},t)$ with respect to $\mathbf{u}_+$ gives the equation for the velocity-difference distribution $g_-(\mathbf{u}_-,\mathbf{r},t)$. If we define the variables as $\mathbf{u}_-=(u_-,v_-,w_-)$, $\mathbf{r}=(r,0,0)$, we can derive the following equation for the lateral velocity-difference distribution:

$$[\partial/\partial t+(1/2)\alpha(t)\partial^2/\partial v_-^2]\,g_\perp(v_-,r,t)=0 \qquad (13)$$

Since this equation is the one-dimensional version of eq. (11), its solution is immediately given from (12) as

$$g_\perp(v_-,r,t)=g_0(v_-,t)=(t/2\pi\alpha_0)^{1/2}\exp[-v_-^2 t/2\alpha_0] \qquad (14)$$

The same physical arguments as those for the solution (12) are applied to the *inertial normal distribution* (14).

In particular, since the distribution $g_\perp(v_-,r,t)$ has to reduce to the delta distribution in the limit of $r\to 0$, its abrupt change should take place more drastically compared with the velocity-sum distribution. This discontinuity is again resolved by taking account of the viscous similarity.

Longitudinal Velocity-Difference Distribution

Following the same process as for the lateral distribution, we obtain the equation,

$$[\partial/\partial t+(1/2)\alpha(t)\partial^2/\partial u_-^2+(1/3)(8u_-+u_-^2\partial/\partial u_-)\partial/\partial r]\,g_\parallel(u_-,r,t)=0 \qquad (15)$$

for the longitudinal velocity-difference distribution. Obviously, this equation permits the *inertial normal distribution*,

$$g_\parallel(u_-,r,t)=g_0(u_-,t)=(t/2\pi\alpha_0)^{1/2}\exp[-u_-^2 t/2\alpha_0] \qquad (16)$$

for large distance r. However, it also has an *inertial range* solution which is obtained from the self-similar equation,

$$(1-(2/9)\zeta^3)K''-(20/9)\zeta^2K'-(16/9)\zeta K=0 \qquad (17)$$

$$\zeta=u_-r^{-1/3}(t^2/\alpha_0)^{1/3},\quad g_\parallel(u_-,r,t)=r^{-1/3}(t^2/\alpha_0)^{1/3}K(\zeta) \qquad (18)$$

The solution $K(\zeta)$ satisfies the inertial-range similarity $u_-\propto r^{1/3}$ and takes asymmetric form with a cusp-like singularity at $\zeta=(9/2)^{1/3}$ and algebraic tails.

CONCLUSION

The *inertial normality* of the velocity distributions in homogeneous isotropic turbulence seems to have been established so far as the one- and two-point statistics. This provides us with good scope for the study of *viscous similarity* of this turbulence and the extension of the present approach to more complex turbulent motions.

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