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NONLINEAR CONVECTIVE PATTERNS IN SPHERICAL RAYLEIGH-BÉNARD SYSTEMS

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Summary Nonlinear thermal convection in a spherical fluid layer in the presence of spherically symmetric gravity, spherical Rayleigh-Bénard convection, is investigated. At the onset of spherical Rayleigh-Bénard convection, there exists the (2l+1)-fold degeneracy of the linear solution, where l is the degree of a spherical harmonics. Nonlinear convection is studied through fully three-dimensional numerical simulations. Several new spherical patterns of nonlinear convection are found. In particular, a steadily drifting pattern in the form of a single giant spiral roll covering the whole spherical surface without defects is discovered for various Prandtl numbers for the first time.

INTRODUCTION

Thermal convection in a spherical fluid layer subject to a spherically symmetric radial gravity force and a spherically symmetric boundary condition, which will be referred to as spherical Rayleigh-Bénard convection, is associated with many natural phenomena in geophysical and astrophysical fluid systems (for example, Schubert, 1979; Zhang and Busse, 1998). Spherical Rayleigh-Bénard convection also represents an extensively studied exemplary fluid system involving the pattern and orientational degeneracies (for example, Busse, 1975; Chossat, 1979).
We consider a Boussinesq fluid spherical shell with constant thermal diffusivity κ, thermal expansion coefficient α and kinematic viscosity ν in the presence of its own gravitational field

$$\mathbf{g} = -\gamma \mathbf{r},$$

where γ is a constant. A traditional heating model (Chandrasekhar, 1961) is adopted, in which the basic temperature gradient,

$$\nabla T_s = -\beta \mathbf{r},$$

where β is constant, assumes a uniform distribution of heat sources. The problem of thermal convection, which was first formulated by Chandrasekhar (1961), is governed by the dimensionless equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + R\Theta \mathbf{r} + \nabla^2 \mathbf{u}, \tag{1}$$

$$P_r \left(\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta \right) = \mathbf{u} \cdot \mathbf{r} + \nabla^2 \Theta, \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

Where $\mathbf{u}$ is the flow velocity and Θ is the temperature perturbation. We have employed the thickness of the spherical shell, $d = (r_o - r_i)$, as the length scale , d^2/ν as the unit of time, and $\beta d^2 \nu/\kappa$ as the unit of temperature fluctuation of the system, which lead to the two non-dimensional parameters, the Rayleigh number R and the Prandtl number P_r.

PATTERNS OF NONLINEAR CONVECTION

In the limit of large aspect ratio $\eta \to \infty$, the mathematical problem of spherical Rayleigh-Bénard convection becomes identical to that of the classical Rayleigh-Bénard convection in an infinitely extended horizontal fluid layer heated from below. The plane-layer Rayleigh-Bénard convection is perhaps the most intensively studied nonlinear system in the understanding of its pattern formation (for example, Veronis, 1959; Cross and Hohenberg, 1993; Bodenschatz, 2000). Thermal convection occurs when the Rayleigh number R reaches its critical value $R_c = 1708$. Near the onset of convection, roll, triangular, hexagonal and square patterns can exist. When aspect ratio η is finite, the effect of spherical geometry plays an essential role and the selection of nonlinear pattern bifurcating from a spherical symmetric basic state poses a complicated and difficult mathematical problem (Busse, 1975; Chossat, 1979). At the onset of spherical Rayleigh-Bénard convection, $R = R_c$, the general linear solution may be written, for example, as

$$u_r = f_l(r) \sum_{m=0}^{m=l} (C_m \cos m\phi + S_m \sin m\phi) P_l^m(\cos\theta), \tag{4}$$

where (r, θ, ϕ) are spherical polar coordinates, u_r is the radial flow, $f_l(r)$ represents a radial eigenfunction, $P_l^m(\cos\theta)$ denotes standard spherical harmonics of degree l and $C_m, S_m, m = 0, 1, ..., l$ are $(2l + 1)$ arbitrary constants. The value of l corresponds to the minimum Rayleigh number required to initiate convection. Equation (4) indicates that there exists the (2l+1)-fold degeneracy of the solution. A complete elimination of the (2l+1)-fold degeneracy by nonlinearity proves to be a mathematically challenging task. The value of l for the onset of spherical Rayleigh-Bénard convection is solely

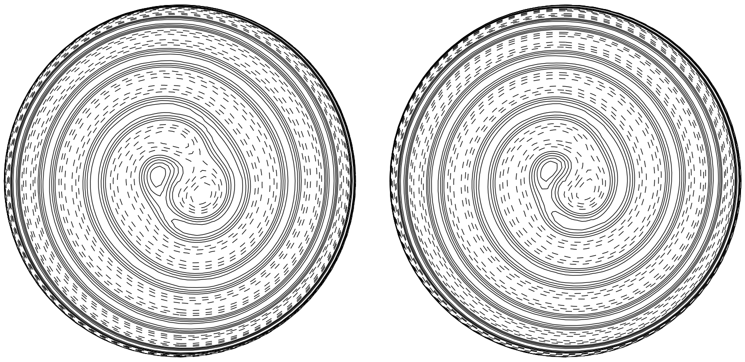

Figure 1. Contours of radial flow u_r at the middle surface of the spherical shell for $\epsilon = (R/R_c) - 1 = 0.28 (R_c = 1712)$ and $Pr = 7.0$. Dashed contours indicate radially inward flow $u_r < 0$ and solid contours correspond to radially outward flow $u_r > 0$. The top (middle) panel shows the contours viewed from the north (south) pole.

determined by the size of aspect ratio η. Our linear analysis reveals that the critical Rayleigh number at $\eta = 41.2$ is given by $R_c = 1712$ with $l = 10$, which is slightly larger than that for the plane-layer Rayleigh-Bénard convection $R_c = 1708$ at $\eta \to \infty$. It follows that the convection solution at the bifurcation point has the 21-fold degeneracy.

Our numerical simulation starts with different combination of spherical harmonics as an arbitrary initial condition near the onset of convection for a fixed Rayleigh number $\epsilon = (R/R_c) - 1 = 0.28$. Convection simulations for a moderate $\epsilon = 0.28$ usually takes several viscous-diffusion-time scales to reach a steady equilibrium. We have carried out an extensive numerical simulations using various initial conditions, demonstrating that the final patterns of steady or periodic equilibriums are strongly dependent upon the initial condition used in simulations. In particular, we have discovered a new pattern in the form of a single, giant spiral roll extending from the north pole to the south pole without defects and covering the whole spherical surface. The giant spiral roll is slowly drifting with a constant amplitude which is shown in Figure 1. Of course the position of the pole at which the spiral roll starts or ends is arbitrary because of orientational degeneracy. We shall also report many other patterns of nonlinear equilibriums obtained using different initial conditions at various Prandtl numbers.

CONCLUSION

This is the first time that a single, giant, perfect spiral roll is found in the problem of spherical Rayleigh-Bénard convection. The discovery of the giant spherical spiral roll suggests the importance of plane asymmetric coefficients S_m in equation (4). In fact, the plane symmetry assumption used in the previous theories excludes the possibility of convection patterns in the form of a single giant spherical spiral roll shown in Figure 1.

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