XXI ICTAM, 15–21 August 2004, Warsaw, Poland

A NEW THEORY FOR CONVECTION IN RAPIDLY ROTATING SPHERICAL SYSTEMS

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Summary Thermal convection in rapidly rotating, self-gravitating Boussinesq fluid spherical systems is a classical problem and has important applications for many geophysical and astrophysical problems. The convection problem is characterized by the three physical parameters, the Rayleigh number R, the Prandtl number P_r and the Ekman number E. This paper reports a new convection theory in rapidly rotating spherical systems valid for $E \ll 1$ and $0 \leq P_r < \infty$. The new theory units the two previously disjointed subjects in rotating fluids: inertial waves and thermal convection. Both linear and nonlinear properties of the problem will be discussed.

INTRODUCTION

Thermal convection in rapidly rotating, self-gravitating Boussinesq fluid spherical systems driven by a uniform distribution of heat sources has been extensively studied. There are two major reasons why much attention has been attracted to this problem: it has important applications for many geophysical and astrophysical problems and it provides a fundamental understanding of general dynamics of rotating fluids.

The classical convection problem is characterized by the three physical parameters, the Rayleigh number R, the Prandtl number P_r and the Ekman number E. In their seminar papers, Roberts (1968) and Busse (1970) established a local asymptotic theory for the onset of convection in the limits $E \to 0$ and $P_r/E \to \infty$. The Roberts-Roberts local theory assumed the following asymptotic laws

$$\frac{\partial}{\partial s} \sim \frac{1}{s}\frac{\partial}{\partial \phi} = O(E^{-1/3}),\ \frac{\partial}{\partial z} = O(1),\ R_c = O(E^{-1/3}) \text{ as } E \to 0 \text{ and } P_r/E \to \infty, \quad (1)$$

where (s, ϕ, z) are cylindrical polar coordinates with the rotation axis in the z-axis. Progress on the asymptotic theory in the limits $E \to 0$ and $P_r/E \to \infty$ for the onset of convection was made by Jones et al. (2000) (see also Soward, 1977), who found the asymptotic solution that produces a correct critical Rayleigh number. The central issue in their analysis is to extend the local solution onto the complex s-plan in which the phase mixing vanishes.

A different asymptotic theory for the onset of convection in a rapidly rotating sphere was developed by Zhang (1994) for the limits $E \to 0$ and $Pr/E \to 0$. It was shown that convective motions are at leading order represented by single inertial wave that has the simplest structure along the axis of rotation. Buoyancy forces appear at next order to drive the inertial wave against the weak effects of viscous damping. It was recognized that the localized convective motion spreads out with decreasing P_r, obeying the following asymptotic laws

$$\frac{\partial}{\partial s} \sim \frac{1}{s}\frac{\partial}{\partial \phi} = O(1), \qquad \frac{\partial}{\partial z} = O(1),\ R_c = O(E) \qquad \text{as} \qquad E \to 0 \text{ and } P_r/E \to 0. \quad (2)$$

In this limit, an analytical expression for the complete convection solution in closed form was obtained (Zhang, 1994). Our new theory attempts to study the solutions of convection for $0 \leq P_r < \infty$ at $E \ll 1$.

A NEW CONVECTION THEORY

Weakly nonlinear solutions of convection can be expressed as

$$\mathbf{u} \quad = \quad \epsilon(\mathbf{u}_0 + \mathbf{u}_0^*) + \frac{\epsilon^2}{E} U(s)\hat{\phi} + ... \quad (3)$$

where ϵ is the amplitude of convection, $\mathbf{u}$ is the flow velocity, $U(s)$ is the nonlinear mean flow, and F^* denotes the complex conjugate of F. The velocity boundary conditions assumed in this paper are stress-free and impenetrable, which give

$$\frac{\partial(\hat{\phi}\cdot\mathbf{u}_0/r)}{\partial r} = \frac{\partial(\hat{\theta}\cdot\mathbf{u}_0/r)}{\partial r} = \hat{\mathbf{r}}\cdot\mathbf{u} = 0 \quad (4)$$

at the outer bounding spherical surfaces. Our new theory for the convection problem is based on the following three hypotheses.

The first hypothesis: For an arbitrary small but fixed Ekman number $E \ll 1$, the localized convection spreads out quickly in both the $s-$ and $\phi-$directions with decreasing P_r. We thus assume that for $0 \leq P_r < \infty$ at an arbitrary small but fixed Ekman number $E \ll 1$,

$$O(1) \leq \frac{\partial}{\partial s} \leq O(E^{-1/3}), \qquad O(1) \leq \frac{1}{s}\frac{\partial}{\partial \phi} \leq O(E^{-1/3}), \qquad \frac{\partial}{\partial z} \sim O(1). \quad (5)$$

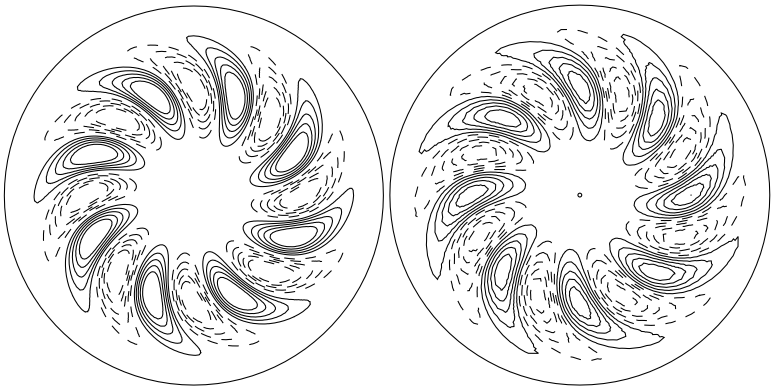

Figure 1. Contours of the radial flow in the equatorial plane. Left panel is the solution derived from the new theory and the solution on right panel is obtained from the fully three dimensional numerical simulations.

A significant consequence is that we must solve partial differential equations. Note that the local theory is concerned with a second-order ordinary differential equation in z.

The second hypothesis: For $0 \leq P_r < \infty$ at an arbitrary small but fixed Ekman number E, we assume that there exists a flow, $\tilde{\mathbf{u}}_0$, that is non-zero only in the Ekman boundary layer on the bounding spherical surface such that

$$\mathbf{r} \cdot \nabla \left[\frac{\mathbf{r}}{r^2} \times (\mathbf{u_0} + \tilde{\mathbf{u}}_0) \right] = 0. \tag{6}$$

The third hypothesis: For an arbitrary small but fixed Ekman number E, we assume that the leading-order velocity of convection can be expressed in the form

$$\mathbf{u}_0 = \sum_N \left[\mathcal{U}_N (\mathbf{U}_N + \tilde{\mathbf{U}}_N) \right] e^{i2\sigma t} \tag{7}$$

where σ is the half-frequency of thermal convection, $\tilde{\mathbf{U}}_N)$ is the boundary layer flow in the Ekman layer, $\mathbf{U}_N$ are nearly geostrophic inertial wave (NGIW) modes for which explicit analytical expressions are now available (Zhang et al., 2001, see also Zhang et al., 2004).

Our analysis demonstrated that agreement between the results obtained from the new theory and fully numerical simulations is excellent. For example, we obtain the critical Rayleigh number $R = 263.5$ from the new theory while the fully numerical simulation gives rise to $R = 264.56$ for $E = 5 \times 10^{-5}$. Based on the linear solution of the problem, we are able to derive an analytic expression for the differential rotation generated by nonlinear interactions of the waves.

CONCLUSION

We show that leading-order convection in a rapidly rotating sphere is either single NGIW mode or a number of coupled NGIW modes modified by viscosity and sustained by thermal buoyancy. The new convection theory unites two previously disjointed subjects in rotating fluids: inertial wave theory and convection theory. This unification furthers our understanding of rotating fluids and opens an exciting line of the future research.

References

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