VORTEX-BASED MODELS OF QUASIGEOSTROPHIC TURBULENCE

Takeshi Miyazaki, Qin Wei, Yingtai Li
Department of Mechanical Engineering and Intelligent Systems, University of Electro-Communications, 1-5-1 Chofugaoka, Chofu-shi, Tokyo 182-8585, Japan

Summary A vortex-based model of the quasigeostrophic turbulence is developed, based on the fact that the interactions between vortex structures dominate the dynamics of the turbulence. Each coherent vortex is modeled by an ellipsoid of uniform potential vorticity embedded in a 'locally uniform shear field' induced by other vortices. The equations of motion are derived following the procedure of Hamiltonian moment reduction. The degree of freedom of N interacting vortices is $3N$, and even a two-body system shows chaotic behavior. The validity of the ellipsoidal moment model is assessed by performing direct numerical simulations based on the CASL-algorithm. It is shown that the model captures the merger of co-rotating vortices well but that it fails to predict the robustness of a counter-rotating vortex pair.

N INTERACTING ELLIPSOIDAL VORTICES - ELLIPSOIDAL MOMENT MODEL -

We consider the motion of N interacting ellipsoidal vortices of uniform potential vorticity $q_i : i = 1, 2, \cdots, N$, whose center of vorticity is located at (X_i, Y_i, Z_i). Here, the z-axis denotes the vertical axis. The potential vorticity q_i is uniform inside the i-th ellipsoid, whose principal axes lengths are $\alpha_i, \beta_i, \gamma_i$, respectively. Their orientations are specified by the Euler angles ϕ_i, θ_i, ψ_i. The state (location, shape, and orientation) of each ellipsoid is also specified by the values of 10 moments up to the second order. [1] The equations of motion are expressed, using a Poisson bracket of the cosymplectic matrix of the Lie-Poisson form. The Hamiltonian of the system is expressed as a summation of the self-energy of each vortex and the mutual interaction energy. We introduce an approximation in order to express the mutual interaction, by using only the moments up to the second order. The obtained equations of motion are similar to those of Meacham *et al.*,[2] in which we have summed up the background vorticity, the strain field, and the vertical shear induced at the center of the $i-$th ellipsoidal vortex by the '$j-$th virtual point vortex'.

This Hamiltonian dynamical system has several invariants, of which three are Casimirs; i.e., the total vorticity (vortex volume) of the ith ellipsoid $\hat{\Gamma}_i$, the $z-$coordinate of the center of the ith ellipsoid Z_i, and the vortex-height of the ith ellipsoid. The degree of freedom of each ellipsoid is reduced to three (six independent variables). There are other conserved quantities; i.e., the total energy H, the vorticity center of the whole system P and Q, and the angular momentum L. Here, $H, P^2 + Q^2, L$ are Poisson-commutable invariants. Even a two-ellipsoids system is not integrable, because it has 'six degrees of freedom'.

MERGER OF CO-ROTATING VORTICES

Miyazaki *et al.*[1] investigated the interaction of two co-rotating spheroidal vortices of the same shape, by adjusting the initial distance between the vortices. When two vortices were placed close enough initially, the motion became chaotic, the horizontal distance $D(t) = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$ between two vortices oscillated irregularly with a large amplitude, and a 'merger' of vortices occurred. By defining the 'merger' to be a phenomenon in which the following two conditions were satisfied (i.e., the vertical overlap ($|Z_2 - Z_1| < zh_1 + zh_2$) and the horizontal overlap ($D(t) < V_1^{1/3} + V_2^{1/3}$) with $V_{1,2}^{1/3}$ denoting the 'average radii'), the threshold of the initial distance leading to the merger was determined. The threshold of the merger was almost the same as that of the chaotic motions. Figure 1 shows the region of the initial position of the second vortex $(a = X_2(0) - X_1(0), h = Z_2 - Z_1)$, relative to the first vortex, for the case of $\alpha_{1,2} = \beta_{1,2} = 0.3162, \gamma_{1,2} = 1$. The shadowed region represents the 'defined' merger region. Starting from the region inside the broken curve (model-threshold), a merger according to the above definition occurs over the course of time.

We performed numerical simulations (CASL) in order to assess the validity of predictions based on the model. The open squares in Fig.1 indicate the cases in which the vortices do not merge. The open circles denote the cases in which the merger is observed. The open triangles indicate the cases of intermediate behavior. The vortices merge once, then separate again into two vortices. This occurs because the transiently created vortex is unstable. The boundaries between three zones are represented by two solid lines. The ellipsoidal vortex model works well for the region $h > 0.5$ (large vertical off-set). The model, however, over-estimates the critical distance in the region $h < 0.5$; i.e., when the vortices are placed on nearly the same horizontal plane.

COUNTER-ROTATING VORTEX PAIR

Next, we investigate the motion of a counter-rotating pair of two vortices of the same shape, a so-called 'dipole'.[3] Here a 'dipole' means a counter-rotating pair of ellipsoids with vanishing total vorticity. They are vertically off-set. We consider prolate spheroids with $\alpha_{1,2} = \beta_{1,2} = 0.3162, \gamma_{1,2} = 1$. Both spheroids are vertically standing at the initial time. The initial vortices are placed on the $x - z$ plane and off-set vertically. The model predicts three patterns, depicted in Fig.2, as follows: (1) stable translation in the positive y direction, (2) translation in the negative y direction with large precessions, and (3) singular behavior (tilting down) of both vortices. In region (1), the motion is doubly time-periodic with a main period of 46.2 and the secondary period of 6.18. The translation velocity in the region (1) is positive. In region (2), both the inclination and orientation change non-periodically. In region (3), the most striking thing occurs; i.e., both vortices are stretched infinitely. The gradient h/a of the boundary-line between the regions (2) and (3) is about 1.41. The tilting pair translates along the y axis in the negative direction. Because of this singular behavior, the numerical computation of the ellipsoidal moment model stops in region (3), which is a serious drawback of the ellipsoidal moment model. It is noteworthy that no singular behavior is observed in the symmetric interactions between fatter vortices, such as $\alpha/\gamma >$ about $2/3$.

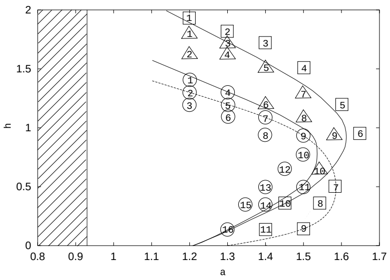

Fig. 1 FigMerger threshold of the co-rotating vortices with $\alpha_{1,2}/\gamma_{1,2} = 0.3162$.

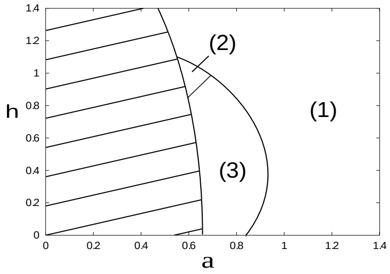

Fig. 2 Three patterns found for a counter-rotating vortex pair with $\alpha_{1,2}/\gamma_{1,2} = 0.3162$.

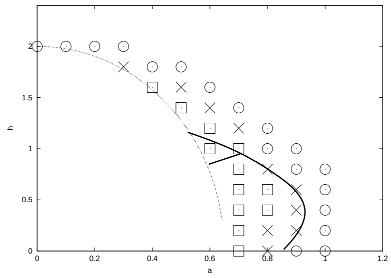

Fig. 3 Pattern map of nonlinear development of counter-rotating vortices.

We performed numerical computations (CASL) corresponding to the above three categories. In case (1), both vortices precess slightly, retaining the ellipsoidal forms, and the energy and enstrophy are conserved. In cases (2) and (3), the vortices tilt largely in the early stage of computation. Later, the vortices emit filaments from the top and bottom and the inclination angles decrease after filamentation (dancing). The singular behavior predicted by the ellipsoidal moment model is not observed; it appears to be circumvented via the dissipative filamentation.

In Fig.3, the nonlinear behavior is summarized as a pattern map in the (a, h) plane. Open circles show the cases of stable translation without dissipation, and crosses represent the cases of vortex-dancing with considerable filamentation. Squares represent the cases in which many satellites remain after filamentation. The solid lines are thresholds predicted by the ellipsoidal moment model. We can see that practically, the model works if the vortices are not off-setted vertically ($h < 1$), for it captures the dissipative processes by giving alarms; i.e., by showing infinite stretching or large precession of one or both of the contour-rotating vortices. When the vortices are largely off-setted ($h > 1$), non-ellipsoidal deformation becomes important, even if the ellipsoidal moment model predicts no singular behavior.

REFINEMENT OF THE ELLIPSOIDAL MOMENT MODEL

Recently, Dritschel and his group proposed an accurate model, [4] in which an ellipsoid is represented by seven point vortices in computing the induced flow field, not by a single point vortex as it is in the ellipsoidal moment model. We have refined the Wire-model (slender limit of the ellipsoidal moment model) following their idea; i.e., by representing a wire-vortex by several point vortices in the integration of the mutual energy. The wire-vortex becomes robuster as the number of point vortices increases. A reasonable choise is ’three’, considering both the accuracy and the computaional time. According to the refined model, the critical merger-distance between co-rotating vortices is also modified in favor of the numerical results.

CONCLUSIONS

A vortex-based model of quasigeostrophic turbulence is developed. The validity (or limitation) of the model is assessed by performing direct numerical simulations based on the CASL-algorithm. The model captures the merger of co-rotating vortices well, but it fails to predict the robustness of a counter-rotating vortex pair. A possible way to refine the model is suggested. We have extracted a workable reset-rule after dissipative events, which enables us to perform ’quasi-turbulence simulations’ based on the refined ellipsoidal moment model.

References

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