COMPLEX -VARIABLE METHODS APPLIED TO FUNCTIONALLY-GRADED ELASTIC PLATE PROBLEMS

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Summary. Exact solutions are obtained to the equations of linear elasticity in a thick elastic plate of inhomogeneous material in which the elastic moduli are specified functions of the coordinate normal to the plane of the plate. These solutions are expressed in terms of four complex potentials that are analytic functions of an in-plane complex coordinate. The solutions admit the specification of the standard resultant force and moment conditions or averaged displacement conditions at the edges of the plate. As illustrations we consider an infinite plate containing a cylindrical hole or through-thickness crack.

There has been a considerable interest in materials that have been deliberately constructed to be inhomogeneous on the macroscopic scale. Such materials may have discontinuous mechanical properties, such as occur in a laminated material, or continuously varying properties as occur in a functionally-graded material. In a series of papers (several of which are summarized and referenced in [1]), Rogers, Spencer and others have developed and applied a procedure for deriving exact solutions of the equations of elasticity for material that are isotropic but whose elastic moduli are general functions of a single space variable. The origins of the method can be found in classical solutions by Michell for moderately thick plates in a state of plane stress. Kaprielian, Rogers and Spencer [2] reformulated and generalized Michell's equations so as to derive exact solutions of the three-dimensional elasticity equations for a material whose elastic constants are any specified functions of a single space coordinate z, which is here taken to be in the direction normal to the surface of a thick flat plate. The dependence of the elastic moduli on z is subject to the usual positive definiteness requirements on the strain energy, but otherwise the Lamé constants λ and μ (or equivalently, Young's modulus and Poisson's ratio) may be any specified functions of z. The dependence of λ and μ on z need not be continuous, so that the case of a laminated material, in which λ and μ are piecewise constant functions of z, is included as a special case. In this formulation any solution of the equations of the plane elastic problem for homogeneous material generates a corresponding *exact* solution of the equations of three-dimensional elasticity for material with the specified inhomogeneity.

A limitation of the method is that, although it generates exact solutions of the field equations in plates which have uniform thickness with traction-free upper and lower surfaces, these solutions are not sufficiently general to match the standard boundary conditions at each point on the edge of a thick plate. However they may be used to satisfy the usual resultant force and moment conditions or averaged displacement conditions at the edge of the plate. Hence these solutions should be regarded as interior solutions in a plate and, for completeness, need to be supplemented by edge boundary-layer solutions.

In plane problems in elasticity, the complex variable methods developed by Muskhelishvili [3] and many others have been used to generate complete solutions of the field equations and have proved very effective in solving boundary-value problems. England and Spencer [4] have applied complex-variable theory to the inhomogeneous thick plate problems outlined above and have shown that it provides a concise and elegant formulation of the problem, and a convenient basis for the solution of a variety of problems. The intention in this presentation is firstly to describe the formulae obtained in [4] and then to describe some illustrative applications to boundary-value problems.

We employ a system of rectangular Cartesian coordinates (x, y, z).in which displacement components are denoted by u, v and w, and the components of the stress tensor σ by $\sigma_{xx}, \sigma_{xy}, \ldots$

We consider a thick plate or slab of linearly elastic material, bounded by the planes $z = \pm h$. The material is isotropic, but may be inhomogeneous in the z direction, so that, in general, λ and μ are specified functions (not necessarily continuous) of z. We also introduce the complex variable $\zeta = x + iy$ and its complex conjugate $\bar{\zeta}$.

Then it was shown in [4] that the three-dimensional displacement field

$$
\begin{aligned}
u(x,y,z) + iv(x,y,z) &= \frac{\kappa_1+1}{\kappa_1-1}\phi(\zeta) - \zeta\overline{\phi\prime(\zeta)} - \overline{\psi(\zeta)} - 2\left(\frac{\kappa_2}{\kappa_1}+z\right)\{\beta(\zeta)+\zeta\overline{\beta\prime(\zeta)}\} - 2z\overline{\alpha\prime(\zeta)} \\
&+ \frac{4}{\kappa_1-1}F(z)\phi\prime\prime(\zeta) - 8\left(\frac{\kappa_2}{\kappa_1}F(z)-B(z)\right)\overline{\beta\prime\prime(\zeta)}, \\
w(x,y,z) &= \alpha(\zeta) + \overline{\alpha(\zeta)} + \bar{\zeta}\beta(\zeta) + \zeta\overline{\beta(\zeta)} \\
&+ \frac{2}{\kappa_1-1}G(z)\left(\phi\prime(\zeta)+\overline{\phi\prime(\zeta)}\right) - 4\left\{\frac{\kappa_2}{\kappa_1}G(z)-C(z)\right\}\left(\beta\prime(\zeta)+\overline{\beta\prime(\zeta)}\right),
\end{aligned}
$$

exactly satisfies the equations of elastostatics in three dimensions for the plate, with zero tractions on the lateral surfaces $z = \pm h$. In these equations, $\phi(\zeta), \psi(\zeta), \alpha(\zeta)$ and $\beta(\zeta)$ are complex potentials and analytical functions of ζ. The coefficients $B(z), C(z), F(z)$ and $G(z)$ are functions of z that depend on the Lamé constants $\lambda(z)$ and $\mu(z)$ and can be evaluated by quadratures when $\lambda(z)$ and $\mu(z)$ are specified. The constants κ_1 and κ_2 can also be evaluated by quadratures when $\lambda(z)$ and $\mu(z)$ are specified.

In general terms the potentials $\phi(\zeta)$ and $\psi(\zeta)$ represent stretching deformations of the plate, and $\alpha(\zeta)$ and $\beta(\zeta)$ are associated with bending deformations. In general these deformation modes are coupled. An exception is the case of a symmetric plate in which $\lambda(z)$ and $\mu(z)$ are even functions of z. In this case $F(z)$ and $C(z)$ become even functions of z and $G(z)$ and $B(z)$ are odd functions of z, and $\kappa_2 = 0$. Then the stretching deformations represented by $\phi(\zeta)$ and $\psi(\zeta)$ uncouple from the bending deformations represented by $\alpha(\zeta)$ and $\beta(\zeta)$, and the two deformation modes can be treated separately.
The stress components associated with the deformation are

$$\begin{aligned}
\sigma_{xx} + \sigma_{yy} &= \frac{4\mu(3\lambda+2\mu)}{\lambda+2\mu}\{\frac{1}{\kappa_1-1}\left(\phi\prime(\zeta)+\overline{\phi\prime(\zeta)}\right) - 2\left(\frac{\kappa_2}{\kappa_1}+z\right)\left(\beta\prime(\zeta)+\overline{\beta\prime(\zeta)}\right)\},\\
\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= 4\mu\{-\zeta\overline{\phi\prime\prime(\zeta)} - \overline{\psi\prime(\zeta)} - 2\left(\frac{\kappa_2}{\kappa_1}+z\right)\zeta\overline{\beta\prime\prime(\zeta)} - 2z\overline{\alpha\prime\prime(\zeta)}\\
&\quad + \frac{4}{\kappa_1-1}F(z)\overline{\phi\prime\prime\prime(\zeta)} - 8\left(\frac{\kappa_2}{\kappa_1}F(z)-B(z)\right)\overline{\beta\prime\prime\prime(\zeta)}\},\\
\sigma_{xz} + i\sigma_{yz} &= 4\mu\{\frac{1}{\kappa_1-1}\left(\frac{dF}{dz}+G\right)\overline{\phi\prime\prime(\zeta)} - 2\left[\frac{\kappa_2}{\kappa_1}\left(\frac{dF}{dz}+G\right) - \left(\frac{dB}{dz}+C\right)\right]\overline{\beta\prime\prime(\zeta)}\},\\
\sigma_{zz} &= 0.
\end{aligned}$$

From these it is straightforward to evaluate the stress resultants and the moments

$$(N_{xx}, N_{yy}, N_{xy}) = \int_{-h}^{h}(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})dz, \quad (M_{xx}, M_{yy}, M_{xy}) = \int_{-h}^{h} z(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})dz$$

in terms of the complex potentials.
Hence we may also calculate the forces and moments that act on a surface generated by an arc C in the ζ-plane in terms of the complex potentials; for example the components (X, Y, Z) of the force on an arc connecting $\zeta = t_0$ to $\zeta = t$ are

$$\begin{aligned}
X + iY &= \frac{4h}{i}[\mu_1\left(\phi(\zeta)+\zeta\overline{\phi\prime(\zeta)}+\overline{\psi(\zeta)}\right) + 2\left(\mu_1\frac{\kappa_2}{\kappa_1}+\mu_2\right)\left(\beta(\zeta)+\zeta\overline{\beta\prime(\zeta)}\right) + 2\mu_2\overline{\alpha\prime(\zeta)}\\
&\quad - \frac{2}{h(\kappa_1-1)}\int_{-h}^{h}\mu F(z)dz\,\overline{\phi\prime\prime(\zeta)} + \frac{4}{h}\left(\frac{\kappa_2}{\kappa_1}\int_{-h}^{h}\mu F(z)dz - \int_{-h}^{h}\mu B(z)dz\right)\overline{\beta\prime\prime(\zeta)}]_{t_0}^{t},\\
Z &= [-\frac{2i}{\kappa_1-1}\int_{-h}^{h}\mu\left(\frac{dF}{dz}+G\right)dz\left(\phi\prime(\zeta)-\overline{\phi\prime(\zeta)}\right)\\
&\quad + 4i\left\{\frac{\kappa_2}{\kappa_1}\int_{-h}^{h}\mu\left(\frac{dF}{dz}+G\right)dz - \int_{-h}^{h}\mu\left(\frac{dB}{dz}+C\right)dz\right\}\left(\beta\prime(\zeta)-\overline{\beta\prime(\zeta)}\right)]_{t_0}^{t}
\end{aligned}$$

where $2h\mu_1 = \int_{-h}^{h}\mu(z)dz,\ 2h\mu_2 = \int_{-h}^{h}\mu(z)zdz.$
Clearly the solution of boundary-value problems reduces to the determination of the complex potentials $\phi(\zeta), \psi(\zeta), \alpha(\zeta)$ and $\beta(\zeta)$. In many simple, but non-trivial, cases, the form of these can be deduced by inspection and use of experience of solving related problems in plane elasticity. For more complicated problems the techniques of complex analysis that have been developed for use in plane elasticity may be applied to determine the complex potentials that are appropriate for specific boundary conditions on the stress resultants, moments and averaged displacements.
To illustrate the theory, we apply the formulae obtained in [4] to investigate the elastic field in a plate containing a cylindrical hole or a line crack through its thickness under a uniform force field at infinity. As might be expected, the elastic field near the cylindrical hole is quite similar to that found in the corresponding plane strain problem, but in the case of a crack the singularity in the stress field near the crack tip is found to be much more extreme than that found in the plane-strain case (see England [5]).
The extension of the theory to include the case of surface loading of the lateral surfaces of the plate is discussed briefly.

References

[1] Mian, M.A., Spencer, A.J.M., *Journal of the Mechanics and Physics of Solids* 46, 2283-2295, 1998
[2] Kaprielian, P., Rogers,T.G., Spencer, A.J.M., *Phil. Trans. Roy. Soc. Lond.* A324, 565-594, 1988
[3] Muskhelishvili, N.I.: Some basic problems of the mathematical theory of elasticity, transl. by J.R.M. Radok, P. Noordhoff, Groningen, 1953
[4] England, A.H., Spencer, A.J.M., *Mathematics and Mechanics of Solids* (to appear)
[5] England, A.H., *Mathematics and Mechanics of Solids* (to appear)