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BOUNDARY-LAYER ANALYSIS OF CHIMNEY STRUCTURES IN MUSHY LAYERS

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Summary The presence of chimneys has a significant effect on the transport of heat and solute through a mushy layer. Coupled nonlinear partial differential equations govern the flow, pressure, temperature, solute concentration and solid fraction in the mushy layer. We use boundary-layer analysis to find a similarity solution and thereby determine predictions of chimney width as well as the fluid, heat and solute fluxes through it.

Background

During solidification of a multicomponent liquid, a morphological instability can occur that results in solid crystals of one component developing, forming a porous medium, with interstitial liquid that is rich in the other components. This is referred to as a mushy layer. When density differences due to concentration variations cause convection, the solid crystals of the porous matrix can be dissolved in the region where the fluid upwells. These cylindrical liquid-filled channels are called "chimneys." Mushy layers and chimneys occur in industrial processes when binary alloys are solidified, and in geophysical situations when saline water is frozen, for example (Worster, 1997).

Main assumptions

We consider a system in which the compositional buoyancy variations are destabilizing, and dominate over thermal buoyancy effects, which we neglect. Since the thermal diffusion coefficient is much larger than the solute diffusion coefficient, we neglect solute diffusion macroscopically. Also we assume that the mushy layer is in thermodynamic equilibrium so that the temperature and concentration are related by the liquidus condition, and we assume that the liquidus relationship can be approximated as being linear. Finally, we neglect changes of density on solidification.

Equations and boundary conditions

We consider the case that the solidifying system is moved at constant speed V through heat exchangers when a steady state has been attained with respect to the heat exchangers. Such conditions are representative of industrial casting processes, and are convenient for mathematical analysis. We denote pressure by p, the Darcy velocity in the moving frame by $\mathbf{u}$, the rescaled temperature by θ and the solid fraction by ϕ. $\Pi(\phi)$ represents the permeability of the mushy layer, and we define the direction of motion to be in the z-direction.

In the mushy layer, we assume that the temperature and solute concentration are related by the liquidus condition. Then the flow, pressure, temperature and solid fraction are determined by the following coupled nonlinear partial differential equations:

$$\begin{aligned} \mathbf{u} = -\mathcal{R}\Pi(\nabla p + \theta \mathbf{e}_z), \quad \text{with} \quad & \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}\cdot\nabla\theta + \mathcal{S}\phi_z \quad = \quad & \nabla^2\theta + \theta_z \\ \mathbf{u}\cdot\nabla\theta + (\phi\theta)_z \quad = \quad & \theta_z + \mathcal{C}\phi_z. \end{aligned} \tag{1}$$

An important parameter in the problem we consider is the Rayleigh number $\mathcal{R} = \beta(C_0 - C_E)g\Pi_0/\nu V$, where $\beta = \rho^{-1}\partial\rho/\partial C$, C_0 is the far-field solute concentration, C_E is the solute concentration at the eutectic point, g is the acceleration due to gravity and ν is the kinematic viscosity. Two other parameters which also appear in the basic equations are the Stefan number $\mathcal{S} = \mathcal{L}/c\Delta T$, where $\mathcal{L}$ is the latent heat per unit mass, c is the specific heat and ΔT is the difference between the liquidus temperature far away and the eutectic temperature; and $\mathcal{C} = (C_S - C_0)/(C_0 - C_E)$, where C_S is the solute concentration in the solid phase, which quantifies how much solute is incorporated on solidification.

The boundary conditions on the chimney wall are found using a lubrication analysis of the flow in the chimney region, as performed by Chung and Worster (2002), plus the condition of marginal equilibrium formulated by Schulze and Worster (1998). We impose that the solution far away from the chimney (i.e. at large x) must decay to a specified far-field temperature distribution of the form cz^b, where c and b are specified positive constants. Another important parameter that enters the boundary conditions is the Darcy number $\mathcal{D} = \Pi_0 V^2/\kappa^2$, where Π_0 represents a typical permeability and κ is the thermal diffusion coefficient.

Solutions

We consider a two-dimensional flow and use the streamfunction formulation. In numerical simulations (Schulze and Worster, 1998) it has been observed that a thermal boundary layer structure develops around the chimney. We focus on this region of the flow. A scaling analysis shows that the mushy layer depth scales as $\mathcal{R}^{-1/3}$, the chimney width scales as $\mathcal{D}^{1/3}/\mathcal{R}^{2/9}$ and the thermal boundary layer width scales as $\mathcal{R}^{-2/3}$ (Schulze and Worster, 1988). We assume that $\mathcal{D}^{1/3}\mathcal{R}^{4/9} \ll 1$ so that the chimney width is small compared to the boundary layer width, and find a solution that is valid in the asymptotic limit that $\mathcal{R} \gg 1$.

Neglecting variations in the permeability and using a lubrication approximation to equations (1), we find similarity solutions in the variables x and z for the streamfunction, the temperature and the solid fraction, that are valid in the thermal boundary layer. The equation that governs the function of the similarity variable is a third order nonlinear differential equation, which is solved by a numerical shooting method. The horizontal velocity $u(x,z)$ varies as $z^{-\frac{1}{2}(1-b)}U(\eta)$, the vertical velocity $w(x,z)$ varies as $z^b W(\eta)$ and the temperature field $\theta(x,z)$ varies as $z^b\Theta(\eta)$, where the similarity variable $\eta = x/z^{\frac{1}{2}(1-\beta)}$ and the functions W and Θ are plotted in Figure 1.

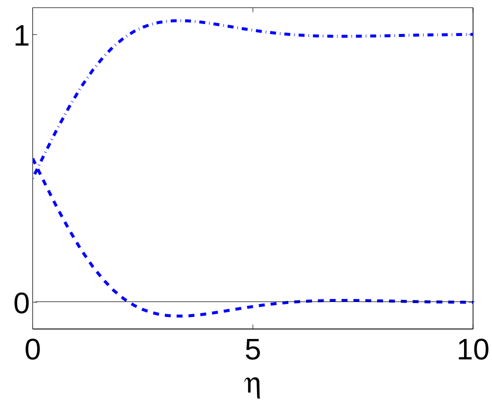

Figure 1. The vertical velocity function $W(\eta)$ (dash line) and temperature function $\Theta(\eta)$ (dash-dot line) as a function of $\eta = x/z^{\frac{1}{2}(1-\beta)}$, for $b = 1$ and $c = 1$.

We also determine numerical results for the fluid flux, the heat and solute flux through the chimney, and the chimney width. The rescaled chimney width varies as $z^{\frac{1}{6}(1-b)}$; the mass flux varies as $z^{\frac{1}{2}(1+b)}$; and the heat flux varies as z^b. Some results for the prefactors, which depend on the far-field boundary condition, are shown in Figure 2. This theoretical study follows from and complements previous numerical studies of chimneys (e.g. Schulze and Worster, 1998; Chung and Worster, 2002). It is also related to studies of the temperature field near a heated plate submerged in a saturated porous medium (e.g. Johnson and Cheng, 1978).

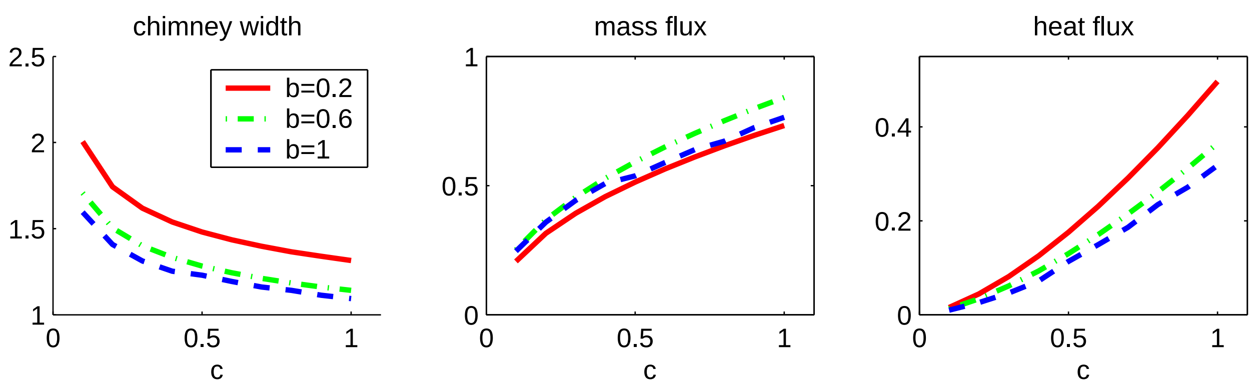

Figure 2. Numerical results for rescaled chimney width, mass flux and heat flux through the chimney. The far-field temperature boundary condition is cz^b.

Conclusions

Our study provides a theoretical prediction of a steady-state chimney. Although the analysis does not incorporate the finite depth of the mushy layer, since it is based on a boundary layer analysis, a suitably truncated solution may be expected to provide a reasonable prediction of experimental results. In addition, the analytical results for a steady-state isolated chimney may be a useful foundation for studies of the dynamics of multiple chimneys.

References

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