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GEOMETRY BASED RATIONAL ENRICHMENT FUNCTIONS FOR TRIANGULAR PLANE ELASTICITY ELEMENT

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Summary In this paper the sides of the triangular plane elasticity *p*-element are allowed to be rational Beziér curves, and the geometry is mapped by the blending function method. If typical hierarchic polynomial shape functions are used, the element is not complete. We present how the element can be made complete by enriching the shape function space with rational functions corresponding to the geometry mapping. A numerical example shows that the enriched element is more efficient than the one without enrichments.

INTRODUCTION

Roots of this research lie in the shape optimization, where the *p*-version of the finite element method has many advantages over the *h*-version. If the element geometry is mapped by the blending function method, the boundary curves of the structure can be the boundary curves of the elements, too. In this case the geometries of the structure and the finite element model are exactly the same, which is desirable especially in the shape optimization. Shyy et al. [1] used parametric polynomial curves as *p*-element sides and stated certain requirements that the element has to meet in order to be complete. When Schramm and Pilkey [2] allowed element sides to be also rational curves, they discovered some numerical inaccuracies. This gave rise to the current research, the aim of which was to improve the *p*-element in such a way that it, even with rational parametric curves as sides, is complete and gives reliable results. In this paper we first recapitulate the blending function method, then present the rational enrichment functions, and finally a numerical example shows how the enriched element performs when compared to the one without enrichments.

MAPPING BY THE BLENDING FUNCTION METHOD

In the blending function method [3], the element geometry is controlled by parametric boundary curves. The element geometry follows exactly the geometry of the boundary curves, and therefore the method is well suited for mapping of the complicated geometries. The blending function method for triangular finite elements was presented by Szabó and Babuška [4], and next we rewrite these mapping functions with slight notational modifications. Element sides are represented by rational Beziér curves

$$\mathbf{R}^i(u) = \left(R^i_x(u), R^i_y(u)\right) \qquad u \in [0,1] \qquad i = 1,2,3, \tag{1}$$

which are directed so that an increasing parameter u corresponds movement anticlockwise around the element. Now, the convention of the side curve directions allows us to define difference functions for side 1

$$D^1_x(u) = R^1_x(u) - (1-u)x_1 - ux_2 \qquad D^1_y(u) = R^1_y(u) - (1-u)y_1 - uy_2, \tag{2}$$

where x_i and y_i are the nodal coordinates. The difference functions for sides 2 and 3 are analogous to those in equation (2). We use common triangular coordinates L_1, L_2 and L_3 to define the three blending functions

$$B^1 = \frac{4L_1L_2}{1-(L_2-L_1)^2} \qquad B^2 = \frac{4L_2L_3}{1-(L_3-L_2)^2} \qquad B^3 = \frac{4L_3L_1}{1-(L_1-L_3)^2}. \tag{3}$$

In the blending function method global coordinates are given by equations

$$x = \sum_{i=1}^{3} L_i x_i + \sum_{i=1}^{3} B^i D^i_x(\xi_i) \qquad y = \sum_{i=1}^{3} L_i y_i + \sum_{i=1}^{3} B^i D^i_y(\xi_i), \tag{4}$$

where auxiliary variables are

$$\xi_1 = \frac{L_2 - L_1 + 1}{2} \qquad \xi_2 = \frac{L_3 - L_2 + 1}{2} \qquad \xi_3 = \frac{L_1 - L_3 + 1}{2}. \tag{5}$$

RATIONAL ENRICHMENT FUNCTIONS

In order for the element be complete, the element shape functions must be able to represent the element geometry. If the element has a rational side, clearly the polynomial shape functions can not represent the geometry. However, we can enrich the shape function space with special rational shape functions, corresponding to the geometry mapping, which make the element complete. Consider again for example the side 1, which we assume to be rational. From the mapping equation (4) the products of the blending and the difference functions vanish on sides 2 and 3, and so they can be used as the side shape functions of side 1. Furthermore, these products are the base of the enrichment side shape functions

$$\hat{N}^1_x = c^1 B^1 D^1_x(\xi_1) \qquad \hat{N}^1_y = c^1 B^1 D^1_y(\xi_1). \tag{6}$$

The factor c^1 scales the values of the enrichment functions to fit for the values of the other shape functions. This prevents numerical problems, since the difference function values may vary largely depending on the element size in the global xy coordinate system. The shape function space always includes the triangular coordinates L_1, L_2 and L_3 (they are the nodal shape functions), and when enriched with the functions (6), it clearly is able to represent the geometry given by the mapping (4). A drawback of the enrichments is that they are different for each rational element side of the FE-model.

A numerical example

We consider a plane stress problem, where the modulus of elasticity is $E = 200$ GPa, Poisson's ratio $\nu = 0.3$, and the sheet thickness $t = 10$ mm. The structure is like a cantilever beam with three holes (see Figure 1), supported from the left end and subject to a stress distribution varying linearly from -30 to 30 MPa at the right end. Each hole is modelled with four second degree rational Beziér curves, each of which has a control polygon with three control points and weights $\{1, 8, 1\}$. Due to the high weight of the middle control point, the curves form almost square holes. The mesh consists of 32 elements, 20 of which are straight sided. The rest 12 elements around the holes each have one very curved and rational side. The mesh is fixed, the curved sides are either enriched or not, and the polynomial degree p of the elements is uniformly increased from 1 to 10. The "exact" results for the problem are from an FE-model with the number of degrees of freedom $N > 259000$. The effect of the enrichments on the performance of *p*-extensions is shown in Figure 2, where various relative errors are plotted against the $\sqrt{N}$. Graphs of the true relative error in energy norm with a) normal and b) enriched elements clearly show the benefits of the enrichments; the error is notably smaller and the rate of convergence at lower values of p is faster. *A posteriori* estimates of the exact strain energy were computed from three consecutive FE-solutions with increasing p, and the relative errors of these estimates with c) normal and d) enriched elements are also shown in Figure 2, where we see again that the enriched elements give more accurate results.

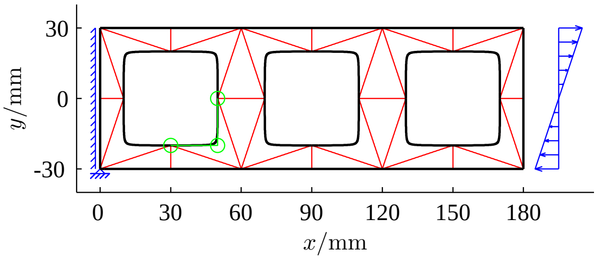

Figure 1. The plane structure (black), its support and loading (blue), the 32 element mesh (red), and an example of a control polygon (green).

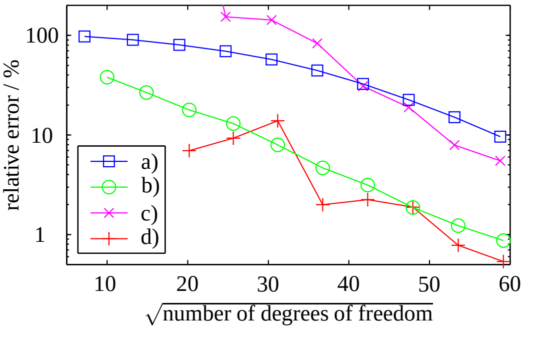

Figure 2. Relative error in energy norm with a) normal and b) enriched elements. Relative error in estimate of exact strain energy with c) normal and d) enriched elements.

CONCLUSIONS

When combined, the *p*-version of the finite element method, rational Beziér curves as element sides, and the blending function method, form an analysis tool suitable for the shape optimization. If the elements are enriched with the rational shape functions as shown in this paper, they will be complete. The numerical example indicates that the enriched element is more efficient and gives more reliable results than the element with no enrichment. A future work might consist of a generalization of the current work for other element types.

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