

## NEW ANALYTICAL APPROACH FOR INVESTIGATION OF NON-STATIONARY DYNAMICS OF MEDIA WITH MOVING INHOMOGENEITIES

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*Summary* Non-stationary processes in various elastic media with moving inclusions like material point bodies, phase boundaries, defects, are under investigation in this paper. The law of motion for an inclusion is considered to be unknown. Small strain approximation is used. The interaction of inhomogeneities with the continuum is described by configurational forces (internal and external). This statement leads to necessity to consider nonlinear (because configurational force is a quadratic quantity) non-stationary problems. A new asymptotical approach for problems of such kind is suggested. The solution of simplest model problem (a point mass moving along a string on an elastic foundation) is given and discussed.

### CONFIGURATIONAL FORCES IN THE CASE OF INFINITESIMAL DEFORMATIONS

In frames of modern continuum mechanics the interaction of media with inhomogeneities like material point bodies, phase boundaries, defects can be described by means of configurational forces. The notion of configurational (material, thermodynamical) force appears as a generalization of notion of a force acting on a defect by Eshelby and crack-driving force (J-integral) by Rice. Configurational forces are quantities, which perform work on velocities of moving sources (defects, cracks, phase boundaries etc) in a continuum. They are useful to describe a dissipation (or, say better, energy exchange with environment) accompanying motion of sources. At the time being the theory of configurational forces is rapidly developing section of continuum mechanics. The state-of-the-art of the theory the reader can find in studies by Gurtin [1], Kienzler & Herrmann [2], Maugin [3]. Below we give a short summary of basic formulae of the theory of configurational forces for the case of infinitesimal deformations. Consider a pure mechanical non-polar medium. In the paper we use the notation of direct tensor calculus, following Lurie [4].

Let  $\mathbf{U}$  be the displacement vector,  $\mathbf{T}$  be the stress tensor,  $\mathcal{U}$  be the strain energy per unit volume,  $\mathbf{I}$  be the identity tensor,  $\rho$  be the mass density,  $\mathbf{b}$  be the bulk external force. The equation for balance of momentum leads to the following relationship

$$\nabla \cdot \mathbf{C} + \mathbf{g} + \mathbf{e} + \mathbf{i} = \mathbf{0}, \quad (1)$$

where we call  $\mathbf{C} = \mathcal{U}\mathbf{I} - \mathbf{T} \cdot \mathbf{U}\nabla$  the configurational stress (the Eshelby tensor),  $\mathbf{g} = -\nabla\mathcal{U} + (\nabla\mathbf{U}\nabla) \cdot \mathbf{T}$  the bulk internal configurational force,  $\mathbf{e} = -\nabla\mathbf{U} \cdot \mathbf{b}$  the bulk external configurational force,  $\mathbf{i} = \rho\nabla\mathbf{U} \cdot \ddot{\mathbf{U}}$  the bulk inertial configurational force. Expression (1) can be transformed into different form

$$\nabla \cdot (\mathbf{C} - \mathcal{K}\mathbf{I}) + \mathbf{g} + \dot{\mathbf{U}}^2 \nabla\rho/2 + \mathbf{e} = \dot{\mathbf{p}}, \quad (2)$$

where  $\dot{\mathbf{p}} = -\rho\nabla\mathbf{U} \cdot \dot{\mathbf{U}}$  is so-called pseudomomentum,  $\mathcal{K}$  is the kinetic energy per unit volume. Quantities  $\mathfrak{C} = \mathbf{C} - \mathcal{K}\mathbf{I}$ ,  $\mathfrak{g} = \mathbf{g} + \frac{1}{2}\dot{\mathbf{U}}^2 \nabla\rho$  are called sometimes the dynamic Eshelby tensor and the dynamic configurational force, respectively.

Introduce now a simple mechanical system, where configurational forces play an important role. This is a migrating control volume. The *migrating control volume*  $\mathcal{P}_t$  is a set of  $\mathbf{r}$  such as  $\mathbf{r} \in \mathcal{P}_t$  if and only if  $\mathbf{r} - \boldsymbol{\ell}(t) = \mathbf{x} \in \mathcal{P}$ . Here  $\mathcal{P}$  is an arbitrary domain fixed in reference configuration. Body  $\mathcal{P}$  consists of different material points at different time instants. For an arbitrary field  $\Phi(\mathbf{r}, t)$  introduce the *time derivative of  $\Phi$  following  $\mathcal{P}_t$* :  $\check{\Phi}(\mathbf{r}, t) = \partial\Phi(\mathbf{x} + \boldsymbol{\ell}(t), \tau)/\partial\tau|_{\tau=t}$ . In fact,  $(\check{\cdot})$  is the derivative with respect to time in co-moving with  $\mathcal{P}_t$  co-ordinate system. One has  $\check{\Phi} = \dot{\Phi} + \mathbf{v} \cdot \nabla\Phi$ , where  $\mathbf{v} = \dot{\boldsymbol{\ell}}$ . Let us formulate the energy balance equation for  $\mathcal{P}_t$ . One can obtain

$$\frac{d}{dt} \int_{\mathcal{P}_t} (\mathcal{K} + \mathcal{U}) dV + \gamma = \int_{\partial\mathcal{P}_t} \mathbf{n} \cdot \mathbf{T} \cdot \check{\mathbf{U}} dS + \int_{\mathcal{P}_t} \mathbf{b} \cdot \check{\mathbf{U}} dV + \mathbf{v} \cdot \int_{\partial\mathcal{P}_t} \mathbf{n} \cdot (\mathcal{K}\mathbf{I} + \mathbf{C}) dS + \mathbf{v} \cdot \int_{\mathcal{P}_t} \mathbf{e} dV \quad (3)$$

Here  $-\gamma$  is the rate of non-mechanical energy exchange with environment of control volume  $\mathcal{P}_t$  ( $\gamma$  is the dissipation),  $\mathbf{n}$  is the external normal to  $\mathcal{P}_t$ . Exactly this relationship allows us to introduce the notions of “force on an inhomogeneity”, “force on a defect” etc. Utilizing the time derivative  $(\check{\cdot})$  one can rewrite (2) into the form, which will be useful when considering dynamic problems concerning moving sources. Let us transform material derivatives in (2) in accordance with definition of  $(\check{\cdot})$ . After calculations one gets

$$\nabla \cdot (\mathbf{C} - \mathfrak{K}) + \mathbf{g} + \mathbf{e} = \check{\mathbf{p}}, \quad \mathfrak{K} = \rho\left(\frac{1}{2}\check{\mathbf{U}}^2\mathbf{I} + (\mathbf{v}\nabla\mathbf{U} \cdot \check{\mathbf{U}} - \mathbf{v} \cdot \nabla\mathbf{U} \cdot \check{\mathbf{U}}\mathbf{I}) + \left(\frac{1}{2}\mathbf{v} \cdot \nabla\mathbf{U} \cdot \mathbf{U}\nabla \cdot \mathbf{v}\mathbf{I} - \mathbf{v}\nabla\mathbf{U} \cdot \mathbf{U}\nabla \cdot \mathbf{v}\right)\right). \quad (4)$$

### A MODEL PROBLEM AND ITS SOLUTION

Consider the classical problem concerning point body moving along the string on Winckler’s foundation [5, 6]. We use the following notation:  $u(x, t)\mathbf{e}_\perp$  is the transversal displacement of a point of the string with a position  $x$  at an instant of time  $t$ ;  $T_0$  is the string tension;  $\rho$  is the density of the string per unit length;  $c = \sqrt{T_0/\rho}$  is the speed of sound (the critical

speed);  $l(t)$  is the position of the point body;  $k_0 = T_0 k > 0$  is the rigidity of foundation;  $\mathbf{f}_0 = T_0(f_{\parallel} \mathbf{e}_{\parallel} + f_{\perp} \mathbf{e}_{\perp})$  is the prescribed external force acting on the point body;  $\mathbf{F}_0 = T_0(F_{\parallel} \mathbf{e}_{\parallel} + F_{\perp} \mathbf{e}_{\perp})$  is the force acting on the string from the point body;  $m_0 = T_0 m$  is the mass of the point body. The square and angle brackets denote the jump and mean value at  $x = l(t)$ , respectively. The governing equations are as follows

$$u'' - c^{-2} \ddot{u} - ku = -F_{\perp} \delta(x - l(t)), \quad \frac{d^2}{dt^2} u(l(t), t) = f_{\perp} - F_{\perp} \quad (5)$$

The motion of the point body along the string is governed by the following law [5, 6]

$$m\ddot{l} = \mathcal{F} + f_{\parallel} \quad \mathcal{F} = F_{\perp} \langle u' \rangle = -(1 - \dot{l}^2/c^2)[u'^2]/2. \quad (6)$$

Here  $\mathcal{F}$  is so-called wave resistance force [5, 6]. The second equation in (6) can be obtained using energy balance equation in the form similar to Eq. (3) formulated for the control volume  $[l(t) - R; l(t) + R]$  with sufficiently large  $R$ . Comparing the general definition of the external configurational force (see above) and the first representation for  $\mathcal{F}$  from (6) one can show that  $\mathcal{F}$  can be considered as external configurational force exerting on the string from the point body. The second representation of  $\mathcal{F}$  follows from one-dimensional equation, which is analogous to Eq. (4). In fact the quantity  $-(1 - \dot{l}^2/c^2)[u'^2]/2$  is exactly the jump of the one-dimensional analogue of tensor  $\mathbf{C} - \mathfrak{R}$ . For simplicity consider zero initial conditions for  $u$ . Thus we have non-stationary (because  $\dot{l}$  is not a constant) nonlinear problem with unknown boundary  $l(t)$ . The nonlinearity is associated with quadratic law for  $\mathcal{F}$ . To investigate this problem we suggest a new analytical approach based on the work by Gavrilov & Indeitsev [7], where the motion of the point body with prescribed law  $l(t)$  was considered. In our approach we utilize the stationary phase method and the method of multiple scales.

**Main supposition:**  $\ddot{l}(t) = O(\epsilon^2)$ ,  $t \rightarrow \infty$ ,  $\dot{l} = O(1)$  — quasi-stationary asymptotics.

1. Suppose  $l(t)$  to be a given function,  $\dot{l} = \text{const}$ . For  $t \rightarrow \infty$  select the set of pronounced frequencies in spectrum of  $u(l(t), t)$ .
2. For  $\ddot{l}(t) = O(\epsilon^2)$  approximate  $u(l(t), t)$  by a sum  $w$  of modes with slowly varying pronounced frequencies.
3. Describe the evolution in time of selected modes using the method of multiple scales.
4. Find out the near-field approximation of wave field in a vicinity of the source using the method of multiple scales.
5. Obtain  $w$  for given  $l(t)$ .
6. Find out  $l(t)$  for given  $f_{\parallel}$ .

Application of this approach to the problem under consideration yields the following results. The law of motion for the point body is

$$(M\dot{l})' = f_{\parallel}, \quad M = m \left( 1 + \frac{\alpha}{(c^2 - \dot{l}^2)^{3/2}} \right), \quad \alpha = \frac{c f_{\perp}^2}{4m\sqrt{k}}. \quad (7)$$

The “effective mass”  $M$  depends on the speed  $\dot{l}$  and tends to infinity when  $|\dot{l}| \rightarrow c$ .

## CONCLUSION

In this paper we demonstrated that the “wave pressure force”, which is known from the beginning of XXth century, can be considered as a configurational force and can be calculated as jump (in the one-dimensional case) or flux of a quantity related to the Eshelby tensor. We guess that this is a quite important fact, since it leads to the similarity for mathematical statements of dynamical problems concerning phase boundaries propagation (in the theory of phase transformations of elastic bodies), moving dislocations etc on the one hand, and classical problems concerning moving loads on the other hand. Furthermore, we suppose that the analytical approach suggested, which works efficiently for moving inertial inclusions, can be applied also in frames of modern fields of mechanics mentioned above. In particular we expect that it can be used for the investigation of dynamical problems concerning phase transformation in one-dimensional elastic bodies (strings [8], beams [9]) and to obtain the equation for motion of dislocation [10] in more compact form.

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