XXI ICTAM, 15-21 August 2004, Warsaw, Poland

SOLVING OF INDIRECT PROBLEMS USING TREFFTZ METHOD

Marek S. Karaś*

Institute of Mathematics, Jagiellonian University, Reymonta 4, PL 30-069 Kraków, Poland

Summary In indirect problems we can observe distribution of certain quantities inside the considered region of a given structure and, after that, knowing this distribution, we can approximately define the whole boundary-value problem [4]. Usually this is done by virtue of minimization of a special form of a functional that is dependent on the searched, unknown boundary values. Thus this leads to a kind of optimization procedure. Some improvements of such a procedure are proposed. These improvements are based on using the generalized Trefftz method (in which the trial functions identically fulfill the given partial differential equations) and the fact that the shape of considered structure remains unchanged. Also a method of direct solving of such kind of indirect problems is proposed. The methods are illustrated in the series of numerical examples.

INTRODUCTION

Typically engineering problem is formulated as a boundary value problem. Because of complexity of such boundary value problems there are used numerical methods to solve them. The generalized Trefftz method is one of such numerical methods. The main idea of the method is to use, like shape functions, functions analytically fulfilled the governing differential equations, so-called T-functions (or Trefftz functions). Any linear combination of T-functions is an exact solution of the governing equations, so coefficients of this linear combination are chosen such that the boundary conditions are satisfied as well as possible.

There are two main variants of the Trefftz method. The first one is the global method in which we obtain one global solution in the whole considered region. In this version, usually, collocation formulation is used [2,3]. The second one is so-called T-elements method [1,5] in which the solution is glued from the solutions in the subregions of the considered regions. In this version, usually, the hybrid formulation with an internal Trefftz field and a frame functions (given on the boundary of elements) is used.

The main advantage of the Trefftz method against another methods such that Finite Element Method or Boundary Element Method is the CPU time that is needed for numerical calculations (in a single solution of a structure). This is especially important when we must repeat such calculations hundreds of thousands times in the loop, for example, of optimisation procedure. The same situation is when we solve indirect problem.

In this paper we consider the following indirect problem. Assume that we have the structure that occupies the region Ω with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Also assume that we have given tractions on the boundary Γ_1 and we do not known the tractions (boundary conditions) on Γ_2. Instead of this we known (from measurement for example) the stresses in the fixed control points $\xi_1,...,\xi_k \in \Omega$. Our problem is to recover tractions (boundary conditions) on Γ_2.

SOLUTION

Main idea of the solution

The main idea of the solution of the above-defined indirect problem is following. We choose any distribution $\hat{t}_2$ of tractions (boundary conditions) on Γ_2, and then we calculate the implicated stresses $\hat{\sigma}_1,...,\hat{\sigma}_k$ in the control points $\xi_1,...,\xi_k$ (in this step we must solve the given boundary problem). And after that we compare $\hat{\sigma}_1,...,\hat{\sigma}_k$ with the given (measured) stresses $\overline{\sigma}_1,...,\overline{\sigma}_k$. The comparison is done (in two dimensional case) by virtue of the calculation of the following functional:

$$I = \sum_{i=1}^{k} \left[\left(\hat{\sigma}_{ix} - \overline{\sigma}_{ix}\right)^2 + \left(\hat{\sigma}_{iy} - \overline{\sigma}_{iy}\right)^2 + \left(\hat{\sigma}_{ixy} - \overline{\sigma}_{ixy}\right)^2 \right]. \tag{1}$$

In the loop of our procedure we try to choose $\hat{t}_2$ such that the value of the functional I is as small as possible. For this purpose a kind of evolutionary algorithm was used, but also other kinds of optimisation algorithm can be used. As an recovered tractions on Γ_2 we take the $\hat{t}_2$ with the minimal value of I.

Trefftz formulation

In the global Trefftz method with collocation formulation we fix the collocation points $\eta_1,...,\eta_r \in \Gamma_2$ and $\eta_{r+1},...,\eta_n \in \Gamma_1$, and, for given boundary conditions we solve the following equation:

$$\mathbf{A} \cdot \mathbf{c} = \mathbf{b}, \tag{2}$$

where $\mathbf{b} = \{t_1^n, t_1^s, ..., t_n^n, t_n^s\}^T$ is a vector consisting of normal and tangential (in two-dimensional case) components of the tractions in the collocations points $\eta_1, ..., \eta_n$, $\mathbf{A}$ is the matrix given by the Trefftz functions $\Phi_1, ..., \Phi_N$ and $\mathbf{c} = \{c_1, ..., c_N\}^T$ is a vector including coefficients of a linear combination of $\Phi_1, ..., \Phi_N$. Eq. (2) is usually solved in the least-square sense, so the solution has the form:

$$\mathbf{c} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}. \tag{3}$$

Thus $\mathbf{c}$ depends on $\mathbf{b}$ linearly. Let $\hat{\sigma}$ be the vector collecting stresses $\hat{\sigma}_1, ..., \hat{\sigma}_k$, then:

$$\hat{\sigma} = \mathbf{B} \cdot \mathbf{c}, \tag{4}$$

where $\mathbf{B}$ is the matrix given by $\Phi_1, ..., \Phi_N$. From (3) and (4) we obtain:

$$\hat{\sigma} = \mathbf{B} \cdot \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}, \tag{5}$$

so $\hat{\sigma}$ also depends on $\mathbf{b}$ linearly.

Improvement of calculations

Let write:

$$\mathbf{b} = t_1^n \cdot \mathbf{e_1} + t_1^s \cdot \mathbf{e_2} + ... t_r^n \cdot \mathbf{e_{2r-1}} + t_r^s \cdot \mathbf{e_{2r}} + \mathbf{b_0} \tag{6},$$

where $\mathbf{e_1}, ..., \mathbf{e_{2r}}$ are elements of the canonical basis of the space $\mathbf{R^{2n}}$ and $\mathbf{b}_0 = \{0, ...0, t_{r+1}^n, t_{r+1}^s, ..., t_n^n, t_n^s\}^T$. Then we can write

$$\hat{\sigma} = \mathbf{C_0} + t_1^n \cdot \mathbf{C_1} + t_1^s \cdot \mathbf{C_2} + ... t_r^n \cdot \mathbf{C_{2r-1}} + t_r^s \cdot \mathbf{C_{2r}}, \tag{7}$$

where $\mathbf{C_0} = \mathbf{B} \cdot \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}_0$, $\mathbf{C}_1 = \mathbf{B} \cdot \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{e}_1, \ldots, \mathbf{C}_{2r} = \mathbf{B} \cdot \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{e}_{2r}$. The formulation (7) allows us to improve significantly the procedure of solving our indirect problem. Indeed, we only one time during the whole process calculate the vectors $\mathbf{C}_0, \mathbf{C}_1, ..., \mathbf{C}_{2r}$ and then inside the loop of the procedure we only evaluate the distribution $\hat{t}_2$ in the collocation points $\eta_1, ..., \eta_{r2}$ (the values $t_1^n, t_1^s, ..., t_r^n, t_r^s$), substitute to formula (7), and then evaluate the functional I. This considerably decreases the CPU time needed for calculations inside the single pass of the loop and of course for the whole process.

Direct solutions

The other idea of solving our problem is to solve, with respect to $t_1^n, t_1^s, ..., t_r^n, t_r^s$, the following equation:

$$\mathbf{C_0} + t_1^n \cdot \mathbf{C_1} + t_1^s \cdot \mathbf{C_2} + ... + t_r^n \cdot \mathbf{C_{2r-1}} + t_r^s \cdot \mathbf{C_{2r}} = \overline{\sigma}, \tag{8}$$

where the vector $\overline{\sigma}$ collects the stresses $\overline{\sigma}_1, ..., \overline{\sigma}_k$. The transition of the tractions $t_1^n, t_1^s, ..., t_r^n, t_r^s$ into the distribution $\hat{t}_2$ gives us the solution of our indirect problem. It can be noted that this is a direct form of solution of the indirect problem.

Numerical examples and conclusions

The efficiency of the proposed method was tested in the series of examples in which the real distribution t_2 of the tractions on the boundary Γ_2 was known. The good convergence of the recovered distribution $\hat{t}_2$ to the distribution t_2 showed that the proposed method works good. Also the improvement given by the formula (7) was tested. The measured CPU time spent for the whole calculation then was decreased about hundred times. The direct method proposed above (solution of Eq. 8) is now tasted. Similar improvements can be done when we use T-elements formulation.

Acknowledgement: Grant KBN 4 T11F 012 23 is gratefully acknowledged.

References

[1] J. Jirousek, A. Wróblewski,: T-elements: State of the art and future trends, Arch. Comp. Meth. Eng. **3**, 4, 323-434, 1996.

[2] Karaś M., Zieliński A.P.: Parametric structural shape optimization using the global Trefftz approach. *J. Theor. and Appl. Mech.*, **2**, 38, 285-296, 2000

[3] Karaś, M.: Application of generalized Trefftz method to optimization of elastic structures. *PhD Thesis*, CUT Dept. of Mech., Cracow, 2001.

[4] Liu G.R., Han X.:Computational inverse techniques in non-destructive evaluation, *CRC Press,* Boca Raton, London, 2003.

[5] Qing H. Q.: The Trefftz Finite and Boundary Element Method, *WITPress,* Southampton, Boston, 2000.