AGMON'S CONDITION FOR INCOMPRESSIBLE ELASTICITY: A VARIATIONAL FORMULATION

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Summary Agmon's condition for incompressible elasticity is recast in variational form. The main result obtained is the natural analog of one found for the compressible case by Mielke and Sprenger. The condition is applied to an assortment of illustrative cases.

Let $\mathbf{X_0}$ be a boundary point of an incompressible, elastic body, at which traction is prescribed. Let $-\mathbf{i_3}$ be the outer unit normal at $\mathbf{X_0}$, $\mathcal{G}_1^{(+)}$ a unit cube with base outer normal $-\mathbf{i_3}$. Let $W(\mathbf{F},\mathbf{X})$ be the stored energy function of the material, $\mathbf{F_0}$ the deformation gradient, and p the reaction pressure at $\mathbf{X_0}$. Then the local second variation condition at $\mathbf{X_0}$ is

$$\int_{\mathcal{G}_1^{(+)}} \mathbb{K}\big[\mathrm{grad}\,\mathbf{w}\big]\cdot(\mathrm{grad}\,\mathbf{w})\mathrm{dV} \geq 0. \tag{V2}$$

for all solenoidal vector fields $\mathbf{w}$ which vanish on $\partial_1\mathcal{G}_1^{(+)}$, the top and sides of $\mathcal{G}_1^{(+)}$. Here,

$$\mathbb{K}\big[\mathbf{G_1}\big]\cdot\mathbf{G_2} := W_{\mathbf{FF}}(\mathbf{F_0},\mathbf{X_0})[\mathbf{G_1F_0}]\cdot(\mathbf{G_2F_0}) + \mathrm{p}\,\mathrm{tr}\big(\mathbf{G_1G_2}\big). \tag{1}$$

For any tangent vector $\mathbf{s}$, define operators $\mathbf{M},\mathbf{N_s},\mathbf{P_s}$ by

$$\mathbf{Ma} := \mathbb{K}\,\big[\mathbf{a}\otimes\mathbf{i_3}\big]\mathbf{i_3}, \qquad \mathbf{N_s a} := -\,\mathbb{K}\,\big[\mathbf{a}\otimes\mathbf{s}\big]\mathbf{i_3}, \qquad \mathbf{P_s a} := \mathbb{K}\,\big[\mathbf{a}\otimes\mathbf{s}\big]\mathbf{s}, \tag{2}$$

and domains $\mathcal{H},\mathcal{D}_\mathbf{s}$ by

$$\mathcal{H}:= \mathrm{W}^{1,2}\big([0,\infty)\big)\otimes \mathrm{W}^{1,2}\big([0,\infty)\big)\otimes \mathrm{W}^{2,2}\big([0,\infty)\big), \qquad \mathcal{D}_\mathbf{s}:= \Big\{\mathbf{z}(\cdot)\in\mathcal{H}\Big|\,\dot{\mathbf{z}}\cdot\mathbf{i_3} + i\mathbf{s}\cdot\mathbf{z} = 0\Big\}\ .$$

Define a functional $\mathcal{J}_\mathbf{s}^\Lambda[\cdot]$ on $\mathcal{D}_\mathbf{s}$ by

$$\mathcal{J}_\mathbf{s}^\Lambda[\mathbf{z}] := \int_0^\infty \Big\{\bar{\dot{\mathbf{z}}}\cdot\mathbf{M}\dot{\mathbf{z}} - 2\mathrm{Re}\big(i\,\bar{\dot{\mathbf{z}}}\cdot\mathbf{N_s z}\big) + \bar{\mathbf{z}}\cdot\big[\mathbf{P_s} + \Lambda^2\mathbf{1}\big]\mathbf{z}\Big\}\mathrm{dt}. \tag{3}$$

Theorem 1 The inequality (V2) holds for all smooth complex-valued solenoidal vector-fields $\mathbf{w}$ which vanish on $\partial_1\mathcal{G}_1^{(+)}$ if and only if $\mathcal{J}_\mathbf{s}^\Lambda[\mathbf{z}] \geq 0$ for all $\mathbf{z}\in\mathcal{D}_\mathbf{s}$ and $\Lambda > 0$.

Define quantities

$$2\Delta := \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\mathrm{M}_{\alpha\gamma}\mathrm{M}_{\beta\delta}, \qquad \Delta\mathrm{L}_{\alpha\beta} := \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\mathrm{M}_{\gamma\delta}, \qquad \mathbf{L} = \mathbf{L^T} := \mathrm{L}_{\alpha\beta}\mathbf{i}_\alpha\otimes\mathbf{i}_\beta := \mathbf{Q^2},$$

$$\mathbf{D} := \Big[\mathbf{N_s}^\mathbf{T} + \mathbf{s}\otimes\mathbf{M^{(i)}i_3}\Big]\mathbf{L} - \mathbf{s}\otimes\mathbf{i_3}, \qquad \mathbf{Z} := \mathbf{Q}\big[\mathbf{N_s} + \mathbf{Mi_3}\otimes\mathbf{s}\big],$$

$$\mathbf{Y} = \mathbf{Y^T} := \mathbf{P_s} - \mathbf{Z^TZ} + \mathbf{N_s^Ti_3}\otimes\mathbf{s} + \mathbf{s}\otimes\mathbf{N_s^Ti_3} + (\mathbf{i_3}\cdot\mathbf{Mi_3})\mathbf{s}\otimes\mathbf{s},$$

$$\mathbf{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -i\mathrm{s}_1 & -i\mathrm{s}_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

MAIN RESULT:- If $\Delta > 0$, following three conditions are equivalent:
(i) $\mathcal{J}^0_{\mathbf{s}}[\cdot]$ is nonnegative for all $\mathbf{s}$.

(ii) For each $\mathbf{s}$, there exists $\mathbf{H}$ positive-semidefinite, Hermitian such that if

$$\mathbf{E_H} := \begin{pmatrix} \mathbf{M} & -i\mathbf{N_s} + \mathbf{H} \\ i\mathbf{N_s^T} + \mathbf{H} & \mathbf{P_s} \end{pmatrix},$$

the 5×5 matrix

$$\tilde{\mathbf{E}}_{\mathbf{H}} := \mathbf{A}^{\dagger}\mathbf{E_H}\mathbf{A}$$

is positive-semidefinite .

(iii) For each $\mathbf{s}$, there exists $\tilde{\mathbf{H}}$ positive-semidefinite, Hermitian such that $\tilde{\mathbf{H}}$ solves the algebraic Riccati equation

$$\tilde{\mathbf{H}}\mathbf{L}\tilde{\mathbf{H}} - i\tilde{\mathbf{H}}\mathbf{D}^{\mathbf{T}} + i\mathbf{D}\tilde{\mathbf{H}} - \mathbf{Y} = \mathbf{0}. \qquad \text{(ARE)}$$

If $\Delta = 0$, conditions (i) and (ii) are equivalent.

This result has the same general form as the main result of [2]. Theorem 1 is proved in [1].

Thus, by Theorem 1, (ii), and if $\Delta > 0$, (iii) are each necessary and sufficient conditions for (V2), and the truth or falsity of (V2) may be determined by checking these conditions. This is done for the following examples:
(a) Equibiaxial stretch of a general isotropic material.
(b) General deformations of a neo-Hookean material.
(c) Some particular deformations of a special anisotropic material.

References

[1] G. P. Mac Sithigh. Necessary conditions at the boundary for minimizers in incompressible elasticity. In review.

[2] A. Mielke and P. Sprenger. Quasiconvexity at the boundary and a simple variational formulation of Agmon's condition. *J. Elasticity* **51** (1998) 23-41.