AFFINE SYMMETRY IN MECHANICS OF DISCRETE AND CONTINUOUS SYSTEMS

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Summary Discussed is the status of kinematical and dynamical affine symmetry in mechanics of discrete multiparticle systems and in continuum mechanics.In particular,we develop affinely-invariant geodetic models of mechanical systems with affine degrees of freedom.Discussed is the model of nonlinear elastic vibrations with the dynamics encoded entirely in the kinetic energy form,without any use of potentials or generalized forces.Certain mechanical and microphysical applications ale briefly discussed.

INTRODUCTION

The impact of geometry on the structure of physical laws is a well-known fact.It is commonly known that kinematics and dynamics of discrete multiparticle systems and continuous media have a very important purely geometrical component following from the Euclidean structure of the physical space or pseudo-Euclidean structure of Minkowskian space-time.Surprisingly enough,geometry of the Galilean space-time is much more complicated than the relativistic one,not yet completely understood and full of surprizes.There is an important sector of macroscopic mechanics and fundamental physics which may be simply deduced from the structure of isometry and rotation groups,or,in certain problems – from the structure of Galilei and Poincare groups.It is sufficient to mention various invariance and objectivity principles in continuum mechanics and its constitutive laws [4],general statements concerning the shape of internal forces in isolated systems,and,in fundamental physics,classification of elementary particles and corresponding fields,based on irreducible representations of the rotation and Lorentz groups (homogeneous and inhomogeneous).In macroscopic mechanics of solids this is reflected by the particular role of rigid body mechanics among all constrained mechanical systems [1].In mechanics of microstructured bodies this impact of geometry on mechanics is very striking and convincing in the theory of micropolar Cosserat continua [7],[8],[10].And rigid body is the simplest nontrivial example,just the particular master pattern of a dynamical systems on Lie-group-spaces [1],[2].

AFFINE BACKGROUND OF MECHANICS

But in elementary geometry there exists also a more fundamental sector,namely,the theory of affine and projective invariants.This is everything which has to do with Tales theorem,collinearity,congruence of segments on the same straight-line ,parallelism,the very straight-line concept,etc,but does not use the notions like the length of segments and angular measure.At least from the purely academic point of view of rational mechanics and geometry it is natural to formulate a programme of affine mechanics and physics.Incidentally,the very fundamentals of Newton theory do not assume metrical geometry and are based on puely affine concepts.Euclidean geometry with its measure of lengths and angles ,i.e.scalar product of vectors,becomes essential on the constitutive level,when constructing realistic expressions for interparticle forces or potentials,although even in this respect one can formulate some academic models based on purely affine relationships.It is worth to mention that also in the very fundamental physics there were some attempts of formulating purely affine theory,based on affine invariants and classifying fundamental particles and fields in terms of irreducible representations of the linear and affine groups.On the level of mechanics of extended bodies such an approach suggests us to replace the metrical rigid body concept by that of affinely rigid one,with degrees of freedom fully described by affine group.When using traditional metrical terms we would say that such an object undergoes translations in space,rotations and homogeneous deformations.Systems with purely affine, $GL(3,\mathbf{R})$ – based models of collective and internal degrees of freedom were considered in various problems of microstructured solids.It seems that A.C.Eringen [4] and his followers were pioneers here when formulating the theory of micromorphic continua.The model was also used in macroscopic elasticity,astrophysics and molecular and nuclear dynamics [3],[7],[8],[11] .Obviously,the two latter applications are based on the quantized version of the theory;the nuclear applications have to do with the droplet model of atomic nuclei.Applications in dynamics of molecular crystals are also possible;by the way,the mechanics of micromorphic continua appears as a long-wave limit of molecular crystals dynamics.However,in all such attempts it was only kinematics that was ruled by affine group ,whereas the all existing in literature models violated affine symmetry on the level of dynamics.The novelty of our models is that the dynamics and the structure of kinetic energy is invariant under spatial or material (or both) affine transformations.Therefore,our models may be interpreted in terms of Arnold-Hermann-Marsden analytical mechanics on Lie groups and their homogeneous spaces,with invariant kinetic energy forms.It is very interesting that bounded motions,e.g.,nonlinear elastic vibrations may be dynamically encoded in the very form of the appropriately chosen kinetic energy ,i.e.,metric tensor on the configuration space,even without any use of potentials or generalized forces.This enables one to obtain some rigorous analytical solutions exactly due to the dynamical invariance under affine group. Such models may be interpreted as a discretization of the Arnold,Marsden,Binz,and others description of ideal fluids as right-invariant dynamical systems with the configuration space given by the infinite-dimensional manifold of all volume-preserving diffeomorphisms [2].This is obviously the very drastic discretization,reducing the non-denumerable continuous number of degrees of freedom to a finite one,equal the dimension of the affine,or linear group.The full invariance of our geodetic models on the affine group enables one to obtain ,at least in certain cases,the general solution in an explicit analytical form.More generally,one can obtain some qualitative information about the phase portrait of the system and some family of special solutions,distinguished by the symmetry group of the problem .These solutions,so-called relative equilibria,or relative closed orbits [1],[9],[12] enable one to understand qualitatively the general solution even if we are unable to find it explicitly in terms of elementary functions or well-known special functions.Having in view some microscopic applications we develop also the quantum version of the theory.It may be applied in mechanics of solids,nanostructures and in dynamics of certain essentially microscopic objects which simultaneously may be interpreted in terms of deformable bodies theory.Applications in relativistic microstructured continua are also possible and certain primary results in this field will be presented.In a sense,they develop certain ideas of Eringen,Bressan and others.

ADDITIONAL REMARKS AND CONCLUSIONS

Models of this type are very interesting from the point of view of pure mathematics and rational mechanics,but apparently,they might seem rather academic toy models.We show that it seems to exist a field of non-standard applications in dynamics of such objects like gas bubbles in fluids,fluid droplet inclusions in fluids,defects in solids,and also in certain versions of the droplet model of atomic nuclei.Applications in nanomechanics are not excluded.Some of results we present are also relevant for the theory of one-dimensional chains (one-dimensional multi-body problems) related to various versions of completely integrable lattices (Calogero-Moser,Sutherland and others) [6].We mention also about generalizations based on Hamiltonian systems on other Lie groups like the projective,conformal,and unitary ones.Some affine generalizations of objectivity principles may be also developed.Certain results in micromorphic relativistic continuum theory were obtained.

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