

## EXPLOSIVE INSTABILITIES IN A CLASS OF HYPERELASTIC MATERIALS WITH HIGHER-ORDER GRADIENTS

Igor Dobovšek

*University of Ljubljana, Faculty of Mathematics & Physics  
Jadranska 19, 1000 Ljubljana, Slovenia  
e-mail: igor.dobovsek@fmf.uni-lj.si*

*Summary* We establish a continuum mechanics framework for three-dimensional analysis for the onset of material instability in second-gradient hyperelasticity based on a linear stability analysis of governing field equations. Critical conditions for the formation of two qualitatively different types of instability are discussed. We analyze necessary conditions for the occurrence of exponential growth of infinitesimal disturbances describing a relative motion with respect to the intermediate state of homogeneous deformation. Thus the instability may be considered as a consequence of a singular behavior of constitutive relations and equations of balance.

### INTRODUCTION

The essence of the method presented in the paper is to consider an evolution of exponential disturbances and their growth or decay from homogeneous states which are defined by a set of relations describing isothermal hyperelastic behavior of the material with the second-gradient type of deformation. If the material in the elastic range exhibits a certain amount of non-local interaction with surroundings, the velocity dependence of propagating waves through such a medium depends on the wavelength and it is non-trivial.

Instability develops as a result of explosive growth of imposed disturbance of governing field equations. Special attention must be given to detection of critical conditions leading to the standard and the symmetry breaking bifurcation. We display a derivation of a general dispersion relation of the expanded stability equation and also discuss some particular issues regarding critical conditions leading to the spatial heterogeneity in the evolution of fluctuation.

The question on which we focus is the possibility of existence of a real value of the wave number of imposed perturbation where one or more bifurcation parameters of the characteristic bifurcation set attain extreme. If there exists only a trivial solution with the value of the wave number equal to zero, which implies that the characteristic wavelength goes to infinity, this obviously corresponds to a space-independent situation. Consequently, the dominant mode in the vicinity of the first critical point will induce a homogeneous transition. If the extreme occurs at a non-trivial value of the wave number at a well-defined characteristic wavelength, the resulting dominant mode in the close vicinity of the critical point will be spatially non-homogeneous. So when the system is on the threshold of instability, the disturbance will evolve in time with the specific critical wavelength. The first unstable mode that will be excited is space dependent. Thus the instability will evolve from a spatially homogeneous initial mode into a spatially heterogeneous mode characterized by an intrinsic space scale; the situation also known as space symmetry breaking.

### FOUNDATIONS

Let  $\mathbf{x} = \chi(\mathbf{X}, t)$  designate a smooth motion of the material between its natural reference configuration  $\mathcal{K}_R$  equipped with the set of Lagrangian coordinates  $\mathbf{X} = \{X_K, K = 1, 2, 3\}$  and the actual configuration  $\mathcal{K}_t$  at time  $t$  with the coordinate set  $\mathbf{x} = \{x_k, k = 1, 2, 3\}$ . The deformation mapping  $\mathbf{F} : \mathcal{K}_R \mapsto \mathcal{K}_t$  is defined by  $\mathbf{F} = \chi \nabla_{\mathbf{R}} = \partial \chi / \partial \mathbf{X}$  and has components  $F_{iK} = \partial \chi_i / \partial X_K$ , while  $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$  is the right Cauchy–Green tensor. The second order gradient is defined as

$$\boldsymbol{\kappa} = \mathbf{F} \nabla_{\mathbf{R}}, \quad \{\kappa_{iJK} = \partial^2 \chi_i / \partial X_J \partial X_K\}.$$

We postulate the existence of strain energy function with smooth dependence on both two-point tensor fields. In the sequel  $W(F, \boldsymbol{\kappa})$  represents the strain energy density per unit volume in reference configuration. The strain energy density is assumed to admit an additive decomposition  $W = \Psi(\mathbf{C}) + \Phi(\mathbf{C}, \boldsymbol{\kappa})$  so that

$$\mathbf{P} = \left( \frac{\partial W}{\partial \mathbf{F}} \right)^T = 2 \frac{\partial W}{\partial \mathbf{C}} \cdot \mathbf{F}^T, \quad \{P_{Ik} = 2 \frac{\partial W}{\partial C_{IJ}} F_{Jk}\}, \quad \mathbf{\Pi} = \frac{\partial W}{\partial \boldsymbol{\kappa}}, \quad \{\Pi_{iJK} = \frac{\partial W}{\partial \kappa_{iJK}}\},$$

where  $\mathbf{P}$  is the first Piolla–Kirchhoff stress, and  $\mathbf{\Pi}$  is a new internal stress intensity that may be perceived as the Piolla–Kirchhoff hyperstress.

### FIELD EQUATIONS

To derive the proper form and the structure of governing equations we can basically use the standard approach as developed for example in the classical field theory where one through the proper choice of fundamental field variables in Lagrangian of the system derives the corresponding field equations. The Lagrangian density of the second-gradient field theory of elasticity reads

$$\mathcal{L} = \mathcal{L}(\mathbf{v}, \mathbf{F}, \mathbf{F}\nabla_{\mathbf{R}}) = \mathcal{L}\left(\frac{\partial\chi}{\partial t}, \frac{\partial\chi}{\partial\mathbf{X}}, \frac{\partial^2\chi}{\partial\mathbf{X}\partial\mathbf{X}}\right) = \frac{1}{2}\rho_{\mathbf{R}}(\mathbf{X})\mathbf{v}^2 - W(\mathbf{F}, \boldsymbol{\kappa}).$$

The structure of kinetic energy density remains simple since we do not consider the local influence of microstructure on effects of long-range interaction and dispersion. In this model such effects are induced by second gradients only. Consequently, the corresponding momentum balance in absence of body forces is given as

$$\nabla_{\mathbf{R}} \cdot \mathbf{P} - \boldsymbol{\Pi} \cdot \cdot \nabla_{\mathbf{R}} \nabla_{\mathbf{R}} = \rho_{\mathbf{R}} \dot{\mathbf{v}}.$$

### Perturbation kinematics

By instability we mean instability with respect to small or linear perturbations that can take the shape of a wave. A perturbation of velocity with respect to the current configuration  $\mathbf{x}$  as reference is defined as a relative motion of a body  $\boldsymbol{\xi} = \boldsymbol{\chi}_t(\mathbf{x}, \tau)$ , which designates the position  $\boldsymbol{\xi}$  occupied at time  $\tau$  relative to the spatial position  $\mathbf{x}$  at time  $t$  so that  $\mathbf{F}_t(\mathbf{x}, \tau) = \boldsymbol{\chi}_t \nabla_{\mathbf{x}}$ . The perturbation is characterized by a function  $\Delta\mathbf{v}(\mathbf{x}, t)$  such that

$$\mathcal{K}_t \xrightarrow{\mathbf{F}_t(\tau)} \mathcal{K}_\tau : \quad \mathbf{v}(\boldsymbol{\xi}, \tau) = \mathbf{v} + \Delta\mathbf{v}(\mathbf{x}, t), \quad \Delta\mathbf{v} = \hat{\mathbf{V}} \exp(\omega_R t + i\varphi),$$

where  $\hat{\mathbf{V}}$  is small ( $\|\hat{\mathbf{V}}\| \ll 1$ ) constant amplitude,  $\varphi = k(ct + \mathbf{n} \cdot \mathbf{x})$  is the phase,  $\omega_R = \Re[\omega]$  is the growth rate,  $\omega_I = \Im[\omega]$  is the wave frequency,  $c = \omega_I/k$  is the wave (phase) speed,  $k = 2\pi/\lambda$  is the wave number, and  $\mathbf{k} = k\mathbf{n}$  the wave vector. We seek solutions of the system in terms of normal modes, keeping in mind that the stability analysis is purely local, assuming that the boundaries are far away. In this way we exclude the effects of geometrical and structural instabilities that may emerge from the influence of boundary conditions and close boundaries. Consequently, if  $\Re[\omega] < 0$  then  $\Delta\mathbf{v}$  is exponentially decreasing function with or without oscillatory modulation according to whether  $\Im[\omega]$  is non-vanishing or vanishing. Hence, the basic solution is asymptotically stable. If on the other hand  $\Re[\omega] > 0$ , disturbances tend to grow exponentially and the reference solution is unstable. The regime which separates the two classes of states, stable from unstable, defines the state of marginal stability. By this definition, the marginal state is the state of neutral stability and  $\Re[\omega] = 0$  determines the threshold of instability of the reference state. Two qualitatively different states can be distinguished here. In most cases it is postulated that beside  $\Re[\omega] = 0$  one also has  $\Im[\omega] = 0$ . The eigenvalue  $\omega$  changes its sign by passing through zero. Some authors refer to this type of critical point as a divergence point. Another possibility is the case of Hopf bifurcation where in addition to  $\Re[\omega] = 0$  one also has  $\Im[\omega] \neq 0$ . Consequently, we get a bifurcation into a time-periodic flow. By assuming homogeneous conditions at the reference configuration, by using relations describing incremental kinematics, the linearization of momentum balance and constitutive equations yields

$$i\mathbf{k} \cdot \Delta\mathbf{P} + \Delta\boldsymbol{\Pi} \cdot \cdot \mathbf{k} \otimes \mathbf{k} = \rho_{\mathbf{R}}\omega\hat{\mathbf{V}}, \quad \Delta\mathbf{P} = i\mathcal{A}:\hat{\mathbf{V}} \otimes \mathbf{k} - \mathcal{B}:\hat{\mathbf{V}} \otimes \mathbf{k} \otimes \mathbf{k}, \quad \Delta\boldsymbol{\Pi} = i\mathcal{C}:\hat{\mathbf{V}} \otimes \mathbf{k} - \mathcal{D}:\hat{\mathbf{V}} \otimes \mathbf{k} \otimes \mathbf{k},$$

where the two-point tensor fields  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  are of the 4-th, 5-th, 5-th, and 6-th order, respectively, as a result of mixed partial differentiation with respect to deformation fields  $\mathbf{C}$  and  $\boldsymbol{\kappa}$ . By choosing current placement as reference  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  represent instantaneous or tangent moduli. Expressing the incremental momentum balance in terms of corresponding increments of stress and hyperstress yields the homogeneous system of equations

$$(\mathbf{Q}_R + i\mathbf{Q}_I + \rho_{\mathbf{R}}\omega\mathbf{I}) \cdot \hat{\mathbf{V}} = \mathbf{0},$$

where  $\mathbf{Q}_R$  is the real part of the acoustic tensor of perturbation and  $\mathbf{Q}_I$  is the imaginary part which may vanish under certain symmetry conditions. Due to the complexity of expressions involved both tensors are not given here. A condition for the nontrivial solution of the system yields the characteristic equation which has the major role in stability analysis of the problem. The characteristic equation is given in the form of determinant

$$\mathcal{D}(\omega, \mathbf{k}) = \det[\mathbf{Q}_R + i\mathbf{Q}_I + \rho_{\mathbf{R}}\omega\mathbf{I}] = 0,$$

which in expanded form yields the dispersion relation  $\mathcal{D}(\omega, \mathbf{k}) = 0$  from where we can determine the relationship  $\omega = \omega(\mathbf{k})$ . This relationship is not trivial for majority of second gradient hyperelasticity models and should be carefully examined in connection with the given stability criteria.

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