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RESONANCES AND MIXING IN STOKES FLOWS

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Summary In the present paper we study chaotic advection and mixing in a stationary incompressible Stokes flow between two co-axial counter-rotating cylinders. The velocity field of the flow is a result of a small perturbation of an integrable velocity field. Under arbitrarily small perturbations of a certain kind a domain of chaotic advection within the gap between the cylinders arises. We show that this phenomenon is a consequence of quasi-random changes in the adiabatic invariant of the flow, which occur as a streamline crosses the two-dimensional resonance surface of the unperturbed flow. We derive an asymptotic formula for the change in the adiabatic invariant due to the passages through the resonance and describe the diffusion of the adiabatic invariant due to multiple passages through the resonance.

Problems of chaotic advection in incompressible Stokes flows attract much interest in connection with impurity transport and mixing (see e.g. [1]). It was first shown numerically in [2], that the presence of a special singular surface (separatrix) leads to the appearance of a domain of chaotic dynamics in near-integrable flows. Similar phenomena was later studied analytically, [3], and experimentally, [4]. In the present study we investigate the mixing in steady Stokes flows in the presence of a different kind of singular surfaces: the resonance. As a model problem we chose a stationary flow between two co-axial counter-rotating cylinders, the outer being of a periodically varying diameter (see Fig. 1a). We consider the case when the variations of the diameter of the outer cylinder are small compared with the separation between the cylinders. In the original (unperturbed) system the flow is regular. It possesses two integrals of motion and all the streamlines are closed curves. The perturbation (assumed to be small) consists of two parts: the vertical flow due to the motion of the inner cylinder and the additional flow due to the noncircularity of the outer cylinder. By itself, either perturbation does not break the approximate regularity of the flow. However, when both of the perturbations are present, regardless of how small those perturbations are, the dynamics becomes more complicated. The space is divided into three regions: in the vicinity of either cylinder the flow is still almost regular, while near the middle of the gap streamlines are chaotic and the mixing occurs (see Fig. 1b). The objective of the present paper is to describe the size and the position of the chaotic domain and the rate of mixing.

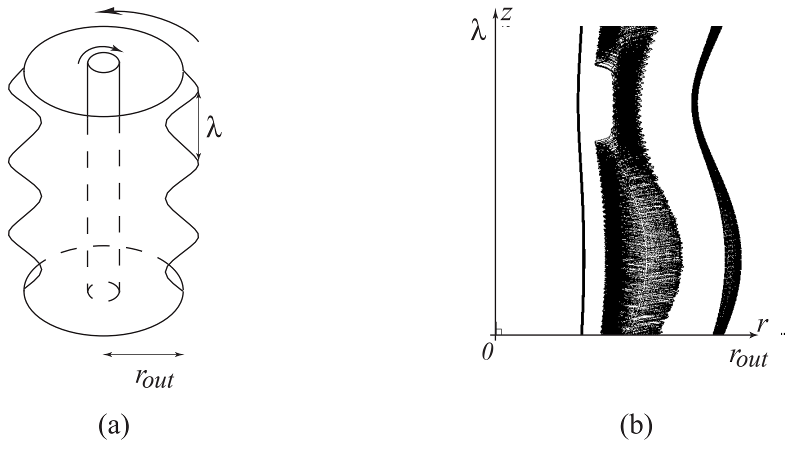

Figure 1. (a) The flow setup. (b) Projection of three streamlines on the (r, z) plane (periodic boundary conditions in z were imposed). The right and left streamlines are regular, the middle one is chaotic.

For a small magnitude of the perturbation, the system possesses two distinct time scales. Streamlines wind fast around the inner cylinder (in clockwise or counterclockwise direction depending on the distance from the axis) while r and

z coordinates of a streamline change much slower. Therefore, in the first approximation we can average equations of motion over the fast (angular) variable and consider the averaged system instead of the exact one. The averaged system gets one new invariant of motion (that is an adiabatic invariant of the exact system) and recovers the regularity of the streamlines.

The exact system follows the averaged one for a very long time everywhere, where the time scales are well separated and averaging is possible. These conditions are satisfied away from a special (*resonance*) surface, where the angular velocity of the unperturbed system vanishes. And it is the vicinity of the resonance surface where the Lagrangian chaos (divergence of initially close streamlines) and mixing are localized. We demonstrate that Lagrangian chaos is a consequence of abrupt changes of the adiabatic invariant, that occur when a streamline crosses the resonance surface.

To describe the processes occurring in the vicinity of resonance we apply the theory developed in [5], [6] for Hamiltonian systems. We show that there are two major phenomena that take place near the resonance. The first is the *scattering on resonance*, a phenomenon that happens on almost every crossing of the resonance surface, but changes a value of the adiabatic invariant and the shape of a streamline only slightly. The second phenomenon is the *capture into resonance*, that changes a value of the adiabatic invariant significantly forcing the exact streamline to abandon the corresponding streamline of the averaged system and to start following the resonance surface. Unlike scattering, capture occurs only for a small measure of the initial conditions. We derive formulas for changes in adiabatic invariants due to both phenomena and compare them with numerical simulations.

The extreme sensitivity of the above phenomena on tiny changes of the initial conditions makes it reasonable (for multiple passages through the resonance) to consider both scattering and capture as probabilistic phenomena. The accumulation of changes of the adiabatic invariant after multiple passages results in destruction of adiabatic invariancy and leads to Lagrangian chaos and mixing. We estimate characteristic time of mixing based on the statistical properties of the resonance phenomena. We show that while the rate of mixing depends on the magnitude of the perturbation, in the first approximation the size of chaotic domain is independent of it, depending only on the extend of variations of the diameter of the outer cylinder. To describe the overall mixing we compute the finite-time Lyapunov exponents and compare the theoretically predicted rate of mixing with results of numerical simulations of the exact system.

References

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