STUDY OF TWO-DIMENSIONAL ELASTICITY ON FGM

Takemasa Seto*, Prof. Tetsu Nishimura*
* Nihon University, Department of Mechanical Engineering, College of Science & Technology, Tokyo, Japan

Summary Functionally Graded Materials (FGMs) are future composite materials, which have non-uniform elastic property over the body. However, the classical theory of elasticity [1] cannot apply to the FGMs because their elastic coefficients are not uniform. Hence, two-dimensional problem for the body of non-uniform elastic modulus will be discussed in this paper. And it is mention that the edge bonded dissimilar materials problem is resolved by the suggested method.

INTRODUCTION

Functionally Graded Materials (FGMs) are developing as future composite materials. This new progressive materials realize the required property of strength or flexibility and a chemical resistance according to the actual applications by changing the property over the body. However, there is no theory to calculate the stress distribution of the structure composed of FGM which has a non-uniform elastic modulus. The classical theory of elasticity is only applicable for the ordinary materials with constant elastic modulus. Hence, the new theory applicable for the body with a non-uniform elastic modulus is demanded.

TWO-DIMENSIONAL PROBLEM

In the classical theory, the differential equation of equilibriums, the compatibility equation and the Hooke's law must be satisfied together with the boundary conditions to determine the stress distribution of two-dimensional problems. Using the stress function ϕ, the compatibility equation, that is the government equation, can be expressed as follows.

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x\partial y}+\frac{\partial^4\phi}{\partial^4 y}=0 \tag{1}$$

Thus the solution of stress distribution of two-dimensional problem reduces to finding the solution of Eq.(1).

COMPATIBILITY EQUATION FOR FGM

In the case of FGMs, although the equations of equilibrium and the compatibility equation are the same with classical theory, the Hooke's law includes the Young's modulus represented by the function of x and y, because the Young's modulus varies over the body. Therefore, using the same procedure as the classical theory, the compatibility equation can be derived as follows.

$$a\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)+2\frac{\partial a}{\partial x}\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y^2}\right)+2\frac{\partial a}{\partial y}\left(\frac{\partial^3\phi}{\partial y^3}+\frac{\partial^3\phi}{\partial x^2\partial y}\right)$$
$$+\frac{\partial^2 a}{\partial x^2}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial^2 a}{\partial y^2}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)+2(1+\nu)\frac{\partial^2 a}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y}=0 \tag{2}$$

Where $a(x)=1/E(x,y)$ and assuming the Poisson's ratio ν is constant. Therefore the solution of stress state of two-dimensional problem with non-uniform elastic modulus is reduced to find this stress function in Eq.(2). It is clear from Eq.(2), the stress distribution is affected by inverse function of the elastic modulus. However, the compatibility equation of non-uniform elastic modulus is too difficult to solve theoretically.

UNI-AXIAL TENSION PROBLEM OF RECTANGULAR PLATE

It is difficult to solve Eq.(2). Here, the uni-axial tension problem of rectangular plate which has the Young's modulus varies only along the x direction (Fig.1) will be considered. Then, Eq.(2) is reduced to the next equation.

$$a\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)+2\frac{\partial a}{\partial x}\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y}\right)+\frac{\partial^2 a}{\partial x^2}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)=0 \tag{3}$$

Here, because of difficulty of solving theoretically, the finite difference method is used in order to calculate this equation in this study. In the numerical calculation, assuming the distribution of Young's modulus is $E(x)=100x+10$. So the value of Young's modulus changes 10 to 110 from end to end. The external forces P=200 are applied to both ends. And the length L=1 and width W=0.25 are used in this numerical calculation.

From the calculated result (Fig.2), it can be seen that the large stress disturbance appears at the limited portion where the Young's modulus is smaller. While, the stress disturbance approaches to uniform around the lager Young's modulus. The second derivative of $a(x)$ takes maximum values at x=0, and decrease abruptly with increasing of x in this calculated condition. So it is concluded that the stress disturbance is strongly affected by the second derivative of $a(x)$.

Fig.1 RECTANGULAR PLATE

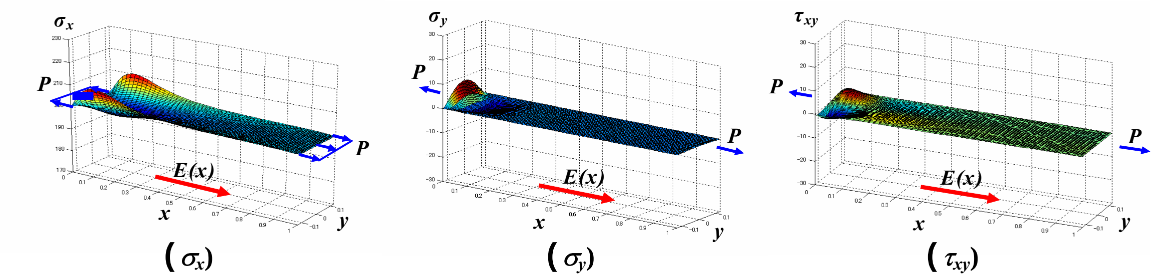

Fig.2 CALCULATED RESULT

APPLICATION TO THE EDGE BONDED DISSIMILAR MATERIALS PROBLEM

The compatibility Equation of non-uniform elastic modulus is also applicable to the edge-bonded dissimilar materials problem when the function of distribution of Young's modulus takes the step function. But it is impossible to use the step function in this calculation because of discontinuous function. Hence, to solve this problem, the sigmoid function Eq.(4) is adopted instead of the step function. The sigmoid function is able to explain the step function when the parameter λ takes to infinity (Fig.3).

$$a(x) = \frac{1}{E(x)} = a_A - \frac{a_A - a_B}{1+\exp(-\lambda x)} \qquad (4)$$

Where, a_A and a_B are inverse of the Young's modulus of material A and B. Fig.4 shows the calculated result in case of λ=30. Here this calculation is assumed that the length L=2 and width W=0.5. The young's modulus of material A and B are 200 and 400 respectively. The external forces P=200 are applied both ends of the plate. The tensile stress σ_x around the joint portion at the free sides takes maximum value in the material A with smaller Young's modulus, and minimum value in the material B with larger Young's modulus. Fig.5 shows the calculated result of difference between maximum and minimum stresses value and divided by external force P. This figure explains the stress disturbance rate of the tensile and lateral stresses. It is clear from this figure that these stress disturbances seems to converge to the certain values for the large number of parameter λ in the sigmoid function. Fig.6 shows difference of the maximum and minimum stress value positions on the x-axis. The stress of σ_x converges to the constant value with increasing λ. Consequently, it can be concluded that the stress distribution of σ_x continues across the bonded portion of the plate.

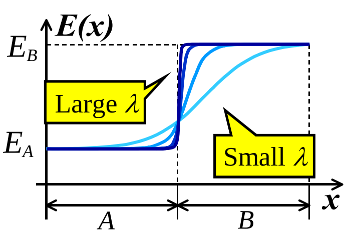

Fig.3 SIGMOID FUNCTION

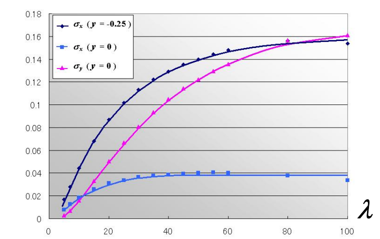

Fig.5 STRESS DISTURBANCE RATE

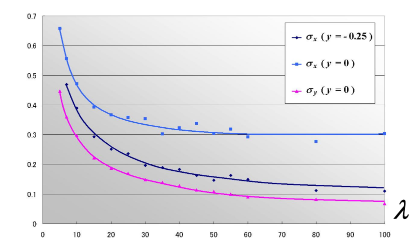

Fig.6 EXTREME VALUE POSITION

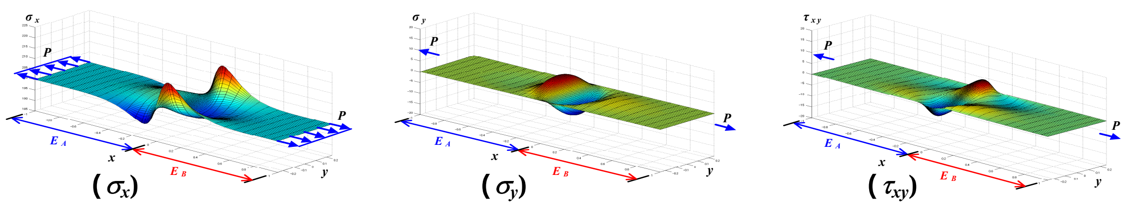

Fig.4 CALCULATED RESULT

CONCLUSIONS

Stress distribution of two-dimensional elastic problem applicable for the non-uniform elastic modulus was analysed by the numerical calculation in this study. And the edge bonded dissimilar materials problem is also applicable to this compatibility equation when the Young's modulus is expressed by the sigmoid function. It is indicated that the limited value of stress exists and the normal stress continuously distributes over the bonded portion.

References

[1] Timoshenko, S. & Goodier, J.N.: Theory of Elasticity (second edition). McGraw Hill, 1951.
[2] Kuranishi, M. : Applied elasticity. Kyoritsu-zensho, 1980.
[3] Omori, M. et al : Proceedings of the 3rd International Symposium on structural and Functionally Gradient Materials, 1994.