RELATION BETWEEN MIXING EFFICIENCY AND GEOMETRICAL PROPERTY OF STABLE MANIFOLDS

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Summary We examine the mixing of fluids associated with the chaotic motion of fluid particles due to the time-periodic flow between two eccentric cylinders caused by the time-periodic slow rotation of these cylinders. We examine the relation between the geomtrical property of the stable manifold of the unstable periodic points of the Poincaré map and the efficiency of the mixing. We find that the maximum stretching rate of fluid elements in a short time is large in the region where the density of the stable manifold is high, and that this stretching rate is small in the region where the curvature of this manifold is large. We also find that small blobs initially located at the region of high density of the stable manifold are mixed well in a short time.

INTRODUCTION

The motion of fluid particles in a flow of an incompressible fluid can be chaotic even if the velocity field of the fluid is steady or time-periodic. The efficient mixing of fluids caused by this chaotic motion is often called the chaotic mixing. There has been many researches on the chaotic motion of fluid particles (Lagrangian chaos) and the chaotic mixing due to two-dimensional time-periodic flows or three-dimensional steady flows since 1980's [1–3].

In the present study, we examine the chaotic mixing due to the time-periodic flow between two eccentric cylinders caused by the time-periodic slow rotation of these cylinders. The main purpose of the present study is to examine what information on the mixing efficiency is obtained from the geometrical property of the stable and unstable manifolds of the unstable periodic points of the Poincaré map which maps the location of a fluid particle onto its location after one period of the flow.

TIME-PERIODIC FLOW BETWEEN TWO ECCENTRIC CYLINDERS

When two eccentric cylinders of sufficiently long length rotate at small angular velocities, we can assume that the flow of an incompressible fluid between these cylinders caused by this rotation is a two-dimensional Stokes flow on the (x, y) plane. Therefore, if we introduce streamfunction ψ satisfying $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, where ($u$,$v$) is the ($x$,$y$) component of fluid velocity $\boldsymbol{u}$, then ψ satisfies

$$\Delta^2\psi = 0, \tag{1}$$

and non-ship condition on the walls of the cylinders, where Δ denotes the Laplacian operator. The solution of this differential equation can be written in the following form :

$$\psi = \Omega_i(t)f_i(\boldsymbol{x}) + \Omega_o(t)f_o(\boldsymbol{x}), \tag{2}$$

where $f_i(\boldsymbol{x})$ and $f_o(\boldsymbol{x})$ are certain functions of spatial coordinate $\boldsymbol{x}$, and $\Omega_i(t)$ and $\Omega_o(t)$ are angular velocities of inner and outer cylinders of radii R_i and R_o, respectively. Here t is the time. It is assumed that the motion of the cylinders is time-periodic and that the outer cylinder alone rotates first by angle πT_o, then the inner one alone by angle $2\pi T_i$, and finally the outer one alone by angle πT_o in one period T. The parameters of this problem are T_i, T_o, radius ratio $\alpha = R_i/R_o$, and the eccentricity $\varepsilon = \ell/(R_o - R_i)$. Here ℓ is the distance between the centers of the cylinders, and then ε ranges from 0 to 1.

When both $\Omega_i(t)$ and $\Omega_o(t)$ are periodic functions of period T, ψ is also a time-periodic function. The time evolution of the location $(x, y) = (X(t), Y(t))$ of a fluid particle is governed by

$$\frac{dX}{dt} = \frac{\partial\psi(X,Y,t)}{\partial Y}, \quad \frac{dY}{dt} = -\frac{\partial\psi(X,Y,t)}{\partial X}. \tag{3}$$

POINCARÉ MAP AND POINCARÉ PLOT

Since ψ is time-periodic with period T, we can define a two-dimensional area-preserving map M that maps $\boldsymbol{X}(t) = (X(t), Y(t))$ onto $\boldsymbol{X}(t+T)$, called the Poincaré map. Furthermore, by plotting the values of $\boldsymbol{X}(nT) = M^n\boldsymbol{X}(0)$ $(n = 0, 1, 2, \cdots)$ for a few initial positions $\boldsymbol{X}(0)$ of fluid particles, we obtain the Poincaré plot.

Since Poincaré map M is area-preserving, the periodic points (including fixed points) of map M are either hyperbolic (saddle type) or elliptic (center type) depending on their eigenvalues. It is commonly observed in the Poincaré plots that if $\boldsymbol{X}(0)$ is sufficiently close to one of the p elliptic periodic points of period p, $\boldsymbol{X}(nT)$ are on one of the closed curves encircling these points for all n. The region around elliptic periodic points where $\boldsymbol{X}(nT)$ of each fluid particle moves regularly on the curves encircling these points is called the regular region or island region.

Moreover, since (3) is a non-antonomous conservative system with respect to two variables, it is possible that this equation has the chaotic solution. Furtheremore, the fluid region between the cylinders is divided into the chaotic region in which each fluid particle moves chaotically and the regular region. The fluid particle starting from the regular (chaotic) region cannot enter the chaotic (regular) region. Therefore, it is better for the efficient mixing to have smaller regular region. Many unstable hyperbolic periodic points of map M are usually imbedded in the chaotic region. Each unstable periodic point has its stable and unstable manifolds. Any fluid particle starting from a location on the stable (unstable) manifold approaches this periodic point as $t \to \infty(-\infty)$.

GEOMTRICAL PROPERTY OF STABLE MANIFOLDS

In the present study, we examine only the case in which almost all the fluid region is the chaotic region. Therefore, the efficient mixing of fluids after many periods is generally expected. However, in the mixing of fluids at the industries, the efficiency of mixing in a relatively small time is also important. Therefore, it may be useful if we can obtain information on this efficiency from the geomtrical property of the stable manifolds of the unstable periodic points of Poincaré map M. Examination of this problem is the main purpose of the present study.

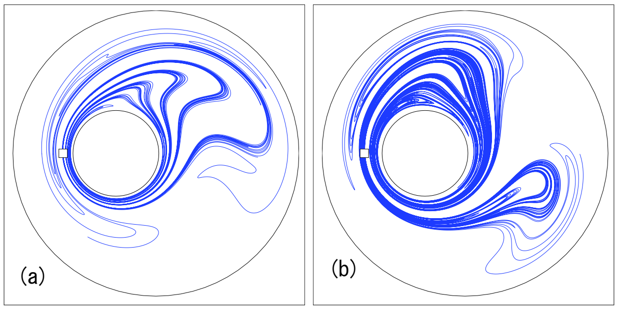

Fig.1 Stable manifolds of unstable fixed points denoted by squares. $\alpha = 0.3, \varepsilon = 0.4$. (a) $T_o = 1.0, T_i = 6.0$, (b) $T_o = 0.5, T_i = 3.0$.

Figure 1 shows examples of parts of the stable manifolds of unstable fixed points. For the parameters of Fig.1, almost all the fluid region is the chaotic region. Therefore, there is no recognizable difference between the Poincaré plots for the parameters of Figs.1 (a) and (b). Although stable manifolds are expected to cover almost all the fluid regions, the manifolds in Fig.1 obtained by inversely mapping for finite times of small line elements directing the stable direction from the fixed points are considerably localized. We find that the density of the manifolds can be closely related to the maximum stretching rate of line elements in a relatively small time. That is, this stretching rate is large at the location where this density is high. From the computation of the evolution of small blobs, we obtain the result that the blob starting from the high-density region is mixed well within a relatively small time, whereas the mixing of the blob starting from the low-density region is bad in a relatively small time, although this blob is also expected to be mixed well after a long time. We also find that the elongation of a line element is suppressed at the location of large curvature of the stable manifold because the direction of the maximum stretching rate at this location in one period is not consistent with the corresponding direction at the points of relatively small curvature on the manifold. There were a few studies on the curvature of material lines in chaotic flows [4–5]. The relation between the results of these studies and the present study also will be discussed.

CONCLUSIONS

The relation between the geomtrical property of the stable manifold of the unstable periodic points of the Poincaré map and the efficiency of the mixing is examined. In conclusion, we find that if a small fluid blob has to be mixed well with a surrounding fluid in a relatively short time, it should be located at the region where the density of the stable manifold is high, and should not be located at the points from which fluid particles move to the region where the curvature of the manifold is high after a short time.

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