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INCLUSION DISPERSION: EFFECTS ON STRESS AND EFFECTIVE PROPERTIES

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Summary A 3D stress analysis method based on Mura's eigenstrains and Eshelby's equivalency principle is proposed. Multiple inclusion interaction is considered, and thus the eigenstrains in each inclusion are no longer uniform. The multiple inclusion problem is solved from the governing elasticity equations to give a set of coupled singular integral equations in the unknown eigenstrains of each inclusion. The set of coupled singular integral equations are rewritten using numerical integration, to give a set of algebraic equations in the unknown eigenstrains. For illustrative purposes the inclusions are dispersed in a cubic arrangement of 3 by 3 by 3 inclusions, thus involving 27 inclusions. 4 different inclusion separation distances are considered and in each of the 4 situations 3 different inclusion stiffnesses are considered.
The obtained stresses from the analysis are seen to be highly influenced by inclusion separation. It is found that the "center" inclusion is shielded by the other inclusions and this shielding effect displays a local minimum as the separation distance changes.

INTRODUCTION

Interaction between heterogeneities such as cracks, flaws, reinforcing fibres or particles has been the subject of research for some years and various methods have been proposed. Influence of heterogeneities on the effective properties of the material have been the subject of intensive study and a vast amount of litterature covers this subject, [1-3]. Especially, the determination of local or effective properties of materials with non-dilute volume fractions of heterogeneities calls for solutions which describes interaction adequately. Since, for 3D problems, no closed form solution of the governing equations exist for more than one inclusion, approximations of some kind are needed. The methods of [4] and [5] are somewhat similar to the present approach, however, in this paper no assumptions are made concerning the distribution of eigenstrains in the inclusions. The present paper solves the system of coupled singular integral equations using numerical integration. The theory presented is general, meaning that the composite may be hybrid, i.e. contain different types of inclusions such as pores and/or particles of different stiffness, but results presented are from spherical inclusions with identical stiffness.
The advantage of the present method is that once the eigenstrains have been obtained it is possible to calculate both local stress/strain fields and obtain informations on the overall effective properties of the considered composite. Furthermore, no assumptions regarding the inclusion dispersion is needed, thus the present theory can be used to quantitatively and qualitatively evaluate various statistical dispersions in terms of e.g. extreme values of stress or overall stiffness.

GOVERNING EQUATIONS

The governing equations for the multi inclusion problem is based on Eshelby's equivalency principle. For inclusion I, the equivalency condition is

$$C^I_{ijkl}(\epsilon^\infty_{kl} + \epsilon^I_{kl}) = C_{ijkl}(\epsilon^\infty_{kl} + \epsilon^I_{kl} - \epsilon^{I*}_{kl}) \tag{1}$$

where C^I_{ijkl} and C_{ijkl} are the stiffness tensor of the inclusions and the matrix, respectively. ϵ^∞_{ij} is the applied strain, ϵ^I_{ij} is the induced strain field due to the presence of inclusion I and ϵ^{I*}_{ij} is the equivalent eigenstrain in inclusion I. The induced strain field, ϵ^I_{ij}, is obtained as the sum of the induced strain field from the inclusion itself and the contributions from all the other inclusions present in the representative volume element, (RVE), as

$$\epsilon^I_{ij} = \int_{V^I} K_{ijkl}(\mathbf{x},\mathbf{x}')\epsilon^{I*}_{kl}(\mathbf{x}')\mathrm{d}\mathbf{x}' + \sum_{R\neq I}^{N} \int_{V^R} K_{ijkl}(\mathbf{x},\mathbf{x}')\epsilon^{R*}_{kl}(\mathbf{x}')\mathrm{d}\mathbf{x}' \tag{2}$$

where V^I and V^R are the volumes of inclusions I and R, respectively. N is the number of inclusions in the RVE, and $\mathrm{d}\mathbf{x}'$ is an infinitesimal volume element centered at $\mathbf{x}'$. The kernel in the integrals, $K_{ijkl}(\mathbf{x},\mathbf{x}')$, contains second order derivatives of Green's function $G_{ij}(\mathbf{x},\mathbf{x}')$ as

$$K_{ijkl}(\mathbf{x},\mathbf{x}') = -\frac{1}{2}C_{pqkl}[G_{ip,qj}(\mathbf{x},\mathbf{x}') + G_{jp,qi}(\mathbf{x},\mathbf{x}')] \tag{3}$$

with

$$G_{ij}(\mathbf{x},\mathbf{x}') = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{(3-4\nu)\delta_{ij}}{|\mathbf{x}-\mathbf{x}'|} + \frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}\right] \tag{4}$$

where μ and ν are the shear modulus and Poisson's ratio of the matrix material.

The above four tensor equations form a system of $6N$ by $6N$ coupled singular integral equations in the unknown eigenstrains. The first integral in Eq. (2) contain a singularity at $\mathbf{x} = \mathbf{x}'$, which must be removed. This is done by rewriting the integral as

$$\int_{V^I} K_{ijkl}(\mathbf{x},\mathbf{x}')\epsilon^{I*}_{kl}(\mathbf{x}')\mathrm{d}\mathbf{x}' = \int_{V^I} K_{ijkl}(\mathbf{x},\mathbf{x}')(\epsilon^{I*}_{kl}(\mathbf{x}') - \epsilon^{I*}_{kl}(\mathbf{x}))\mathrm{d}\mathbf{x}' + \left(\int_{V^I} \mathbf{K}(\mathbf{x},\mathbf{x}')\mathrm{d}\mathbf{x}'\right)\epsilon^{I*}_{kl}(\mathbf{x}) \quad (5)$$

now the first integral on the rigth hand side becomes zero when $\mathbf{x} = \mathbf{x}'$ and the second integral is simply the Eshelby tensor, S_{ijkl}.
Now applying a numerical integration procedure decouples the singular integral equations to a set of algebraic equations in the unknown eigenstrains.

RESULTS AND DISCUSSION

The 3 inclusion stifnesses, E_f, considered are: $E_f = 5E_m$, $10E_m$ and glass particles with $E_f = 73000$ MPa. The Poissons ratios are, $\nu_m = 0.32$ and $\nu_f = 0.3$ and $E_m = 1500$ MPa. The inclusion separation is given in terms of inclusion radius, r, (r =0.01 mm) and the edge-to-edge separation δ, see Fig. 1A. δ in the 4 cases are: $\delta = \frac{r}{10}, \frac{r}{5}, \frac{r}{2}$ and r. The RVE was applied to the 6 different load cases $\sigma^{\infty}_{ij} = 1$ MPa. In each situation the volume averaged stresses for each inclusion, $\langle\sigma^{(I)}_{ij}\rangle$ is calculated. The volume averaged stress for the "center" inclusion, $\langle\sigma^{(14)}_{11}\rangle$ in the situation of $\sigma^{\infty}_{11} = 1$ MPa is shown in Fig. 1B. The figure shows that inclusion separation has a pronounced effect on the volume averaged stress,

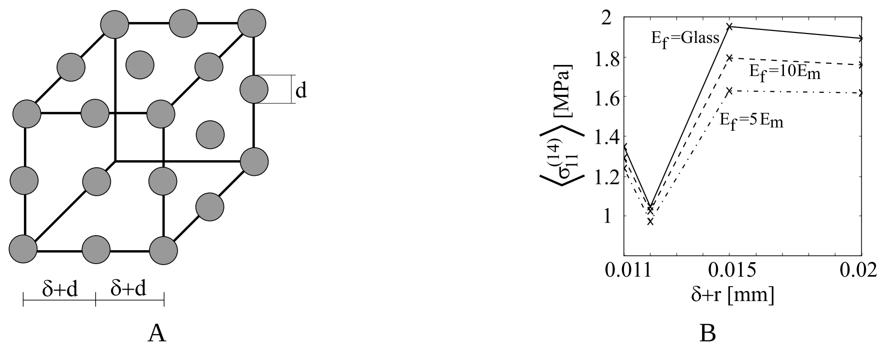

Figure 1. *A) The RVE and B) $\langle\sigma^{(14)}_{11}\rangle$ as a function of inclusion separation, $\delta + r$.*

which is what should be expected. However, the interesting feature displayed by Fig. 1B is that at intermediate inclusion separations there is a minimum of average inclusion stress which suggest that for this type of RVE there is an optimal volume fraction of inclusions, at which the inclusions, in an average sense, do not introduce severe stress concentrations. This may be beneficial for designing composites which are less sensitive to failure. The reason for the behaviour of the graph of $\langle\sigma^{(14)}_{11}\rangle$ is due to shielding effects. At small inclusion separations the center inclusion is situated in an area which shields the inclusion, however, the shielding effect becomes more pronounced for slight increases in separation distance, leading to a decrease in stress. Finally, further increase of the inclusion separation diminish the shielding effect and the average stress increases.

CONCLUSIONS

A 3D stress analysis method based on the theory of eigenstrains and Eshelby's equivalency principle has been shown. Multiple inclusion interaction is considered, leading to non-uniform eigenstrains in the inclusions. Volume averaged quantities are readily obtained and the method may be used for calculating both local field quantities and effective properties. An interesting feature displayed by the volume averaged stress,$\langle\sigma^{(I)}_{ij}\rangle$, of the "center" inclusion of the RVE is that at intermediate inclusion separations there is a minimum of inclusion stress which suggest that for this type of RVE there is an optimal volume fraction of inclusions, at which the inclusions, in an average sense, do not lead to severe stress concentrations. This may be beneficial for designing composites which are less sensitive to failure.

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