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EULERIAN MEASURES FOR LAGRANGIAN STIRRING IN A THERMALLY DRIVEN FLOW

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Summary Lagrangian stirring in a thermally driven rotating annulus is investigated numerically. The stirring is quantified using Eulerian symmetry measures, as well as more commonly used Lagrangian measures. The ability of the measures to identify transport barriers and regions of well and poorly stirred flow is investigated, and space and time averages of the Eulerian symmetry measures are compared to those of the Lagrangian measures for various flow regimes.

Introduction

Tracer transport in fluid flows, particularly atmospheric fluid flows, is commonly studied using analysed wind (velocity), temperature and pressure fields. Tracer trajectories are integrated forward in time, using the velocities defined on a grid, and interpolating them to the required point at each step of the integration. This technique has provided many useful insights, particularly in the stratosphere, where the Lagrangian motion can be almost entirely determined by the large scale wind structure. However, integrating many such trajectories is computationally expensive and is less accurate in other areas of the atmosphere, particularly where the wind fields are known only on a sparse grid. It is therefore of benefit to discover connections between Eulerian velocity fields and the Lagrangian structure produced by such fields without the need to carry out detailed integrations of tracer transport. Additionally, such connections are arguably of interest at a fundamental level, and may reasonably be expected to find applications in many areas of fluid mechanics.

Recently [1] discovered that integrated values of a suitably chosen Eulerian measure of the local three dimensionality of the phase space were strongly correlated with the extent of stirring in the flow, as measured by an effective diffusion coefficient. An objective of our work is to evaluate Eulerian symmetry measures for thermally driven rotating annulus flows and to compare them with Lagrangian measures of stirring, such as finite time and finite scale Lyapunov exponents.

The Flow and Methods

The flow considered is that of a viscous, thermally conducting fluid in a differentially heated rotating annulus, with rigid sidewalls and lower endwall, and a stress-free upper endwall. The flow is held to be axisymmetric but the sidewall heating is time dependent. The two numerical codes used are a Navier Stokes field model, and a particle path tracking code similar to that used by [1]. Because the Navier Stokes model is axisymmetric, the velocity fields produced by it are constrained to be two dimensional.

The velocity field in the r-z plane, (u, w), is used to find tracer trajectories in r and z using the particle path tracking code, which integrates the equations $(\dot{r}, \dot{z}) = (u, w)$ for a predetermined starting position for each tracer. The integration is done using a 2nd order Runge-Kutta routine in either direction. For each integration step, the velocity fields are interpolated to the position of the tracer, and to the intermediate point required for the Runge-Kutta integration, from the nearest grid points using a bicubic interpolation in space and a linear interpolation in time.

In the absence of time dependence in the temperature forcing, the flow does not produce chaotic tracer trajectories, since the number of phase space dimensions is two. Introducing the time dependence increases the number of phase space dimensions to three (two space and time), so it is possible to observe chaotic tracer trajectories and hence enhanced stirring.

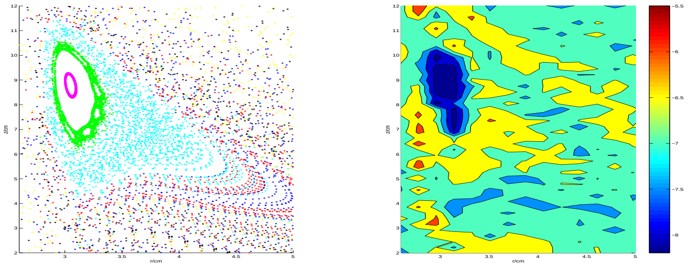
A motivation for seeking Eulerian measures for enhanced stirring comes from a consideration of the Hamiltonian formulation of the governing equations of motion. For example, a two dimensional incompressible flow can be written in terms of a streamfunction, which can be considered to play the role of the Hamiltonian. If the streamfunction is time independent then, the flow is integrable, whereas if the streamfunction is time dependent then a function F satisfying $\frac{\mathrm{d}F}{\mathrm{d}t} = 0$ is required for integrability. If it can be written $\frac{\mathrm{d}F}{\mathrm{d}t} = \phi(\mathbf{V})$ where $\mathbf{V}$ is the velocity field, then ϕ provides a measure of departure from dynamical symmetry. Similarly, the time dependence of the streamfunction provides a measure of departure from geometrical symmetry. For a time dependent two dimensional flow field, then, regions of the flow where ϕ or $\frac{\partial \psi}{\partial t}$ is zero will give rise to integrable trajectories. Physically, tracer particles in regions of zero ϕ or $\frac{\partial \psi}{\partial t}$ will be constrained to two dimensional surfaces, and will therefore not be well mixed in these regions. The value of these quantities may therefore be expected to provide some idea of how well mixed tracers advected due to the flow will become in a given region, since, for lower values, trajectories will be more closely constrained to two dimensional surfaces.

While studying the axisymmetric annulus flows, some Lagrangian measures of mixing have been identified, mainly to provide a ‘control parameter’ to which to compare the Eulerian symmetry measures. However, it is interesting to investigate how well these Lagrangian measures themselves can quantify how well mixed the flow is in different regions, for both the axisymmetric and the non-axisymmetric cases.

Results

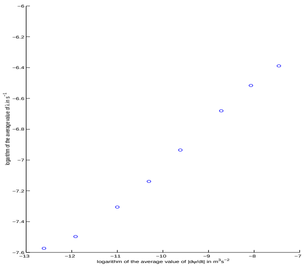
Box counting dimensions, finite time Lyapunov exponents and finite scale Lyapunov exponents were calculated for various flow regimes, with a rotation rate of $1\,\mathrm{rad\,s^{-1}}$ and a mean temperature difference between the inner and outer wall of $4\,\mathrm{K}$, and various forcing amplitudes and frequencies. These Lagrangian measures were found to predict poorly mixed regions, and transport barriers in particular, rather well. A contour plot of the finite scale Lyapunov exponent λ, and the Poincaré section for the same region of the flow, are shown in figure 1. The forcing amplitude and frequency are $1\,\mathrm{K}$ and $2\pi/100\,\mathrm{s}^{-1}$ respectively. The exponent is calculated by taking two tracers a distance $x(0)$ apart, and integrating their trajectories until they have reached a separation $x(\tau) = Rx(0)$. The exponent is given by $\lambda(r,z) = \frac{1}{\tau}\ln R$, where τ is the time at which the separation reaches $Rx(0)$ and (r,z) is the initial position of one of the tracers. The value of R used in the figure is 1000.

Figure 1. Comparason of Poincaré section and contour plot of logarithms of the value of λ in s^{-1}, for a region of the flow



This lyapunov exponent was calculated for eight different flow regimes, all with forcing frequency $\pi/100\,\mathrm{s}^{-1}$ and various amplitudes. The exponent was averaged over the phase space for each flow regime, to give an average value $\bar{\lambda}$. This is compared with the corresponding average value of the geometrical symmetry measure $\frac{\partial\psi}{\partial t}$, which measures the extent of the time dependence of the Hamiltonian (the streamfunction) governing the flow. A log-log plot of the two measures is shown in figure 2 and there is a slope of about 0.24, suggesting that $\bar{\lambda} \propto \left(\overline{\left|\frac{\partial\psi}{\partial t}\right|}\right)^{0.24}$. The geometrical symmetry measure thus seems to be able to predict the extent of the mixing in a given flow as a whole, when the amplitude of the forcing is varied with constant frequency of forcing.

Figure 2. Correlation between logarithms of geometrical symmetry and finite scale Lyapunov exponent



Conclusions

The Lagrangian measures accurately predict the locations of transport barriers.The Eulerian symmetry measures seem to provide a way of efficiently predicting the broader mixing characteristics of a flow, although they are not sufficiently sharp to predict the exact locations of transport barriers.
The next part of this study will be to look at fully three dimensional thermally driven rotating annulus flows, which will provide an opportunity to evaluate the methods in the context of a more dynamically consistent flow regime. It will also be possible then to compare the results with laboratory experiments.

References

[1] King G. P., Rowlands G., Rudman M., Yannacopoulos, A. N.: Predicting chaotic dispersion with Eulerian symmetry measures: Wavy Taylor-vortex flow. *Physics of Fluids* **15**(9):2522–2528,2001