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SPREADING OF CHARGED MICRODROPLETS

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Summary We consider the spreading of a charged conducting droplet on a flat dielectric surface. Two forces drive the spreading: surface tension and electrostatic repulsion. By using the lubrication approximation we derive a fourth order nonlinear partial differential equation that describes the evolution of the height profile. We find that the equation has a two-parameter family of selfsimilar solutions. Some of the solutions are explicitly computed while the other solutions are studied numerically. We show that the solutions have moving contact lines and the radius of the drop is a power law of time with exponent one-tenth. We also construct explicit solutions corresponding to non-circular drops, whose interfaces are ellipses with constant focal length.

MAIN RESULTS

The interaction between a droplet and an electric field has received intense attention in the last few years in connection to its potential technological applications in microfluidics, inkjet printing and electrospray ionization. Here we study spreading drops placed on a dielectric plane surface in the absence of any other external charges or conductors. In particular, we are interested on the influence of the electric charge on the spreading rate and the shape of the drop.
We derive a model based on the thin film approximation and find the following results:

1. For circular drops, there are self-similar solutions that describe the advanced stage of the spreading. We find a two-parameter family of compactly supported self-similar solutions showing that the radius $a(t)$ of the drop is given by a power law of the form
$$a(t) = \zeta(X, Z)\left(\frac{V^3\gamma}{\mu}t\right)^{1/10}, \qquad (1)$$
where μ is the viscosity, γ the surface tension coefficient, V the volume of fluid, t is the time measured since the beginning of the spreading and X, Z are dimensionless parameters.

2. One of these parameters,
$$X = \frac{\varepsilon_0}{V\gamma}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2, \qquad (2)$$
where Q is the electric charge of the drop and ε_0 is the permittivity of the gas above the drop, is identical up to a constant to Lord Rayleigh's fissibility parameter (cf. [2], [1]) and measures the ratio between electrostatic and capillary forces controlling the dynamics. The ratio between these forces does not depend on the radius of the drop and remains constant through the whole spreading process. The other parameter Z controls the speed of the spreading and the shape of the drop.

3. One of these solutions is computed explicitly. In this solution the capillarity does not affect the shape of the drop. The other solutions, whose drop shapes are affected by capillarity, are computed numerically by solving a third order boundary value problem. The self-similar solutions have two possible different shapes: convex drops and drops with an annular bump around the center of the drop.

4. We construct explicit solutions for drops with elliptical interfaces, describing the spreading of a liquid initially concentrated on a finite segment. The subsequent shapes of the interface are a family of confocal ellipses that asymptotically approach a circle.

LUBRICATION APPROXIMATION AND SELF-SIMILAR SPREADING

We shall assume that the drop is thin enough to allow the use of the so-called "lubrication approximation". That is, if h_0 is the height of the drop and a its radius, then $h_0/a \ll 1$. Under this approximation, the velocity field is mostly horizontal and its derivatives with respect to the vertical direction are dominant respect to the derivatives on the other directions.
Let the free surface of the drop be described by the graph $z = h(x, y, t)$. If we impose the no-slip boundary condition at the substrate ($z = 0$) as well as zero tangential stresses at the free surface ($z = h$) then it is well know (cf. for instance [3]) that, for a Newtonian fluid, the velocity averaged on the vertical direction is given by:
$$\overline{v} = -\frac{1}{3\mu}h^2\overline{\nabla}p, \qquad (3)$$
where $\overline{\nabla}p$ is the gradient of the pressure at (x, y), which under the given approximation is independent of z. The equation for the interface follows from mass conservation and reads as:
$$h_t + \overline{\nabla}\cdot(h\overline{v}) = 0\,, \qquad (4)$$

At the free surface the normal stress balance yields the following relationship between pressure, electrostatic stress and capillary pressure:

$$p = -\sigma E_n + \gamma k, \tag{5}$$

where σ is the induced surface charge, E_n is the component of the electric field normal to the surface ($E_n = \overline{E} \cdot \overline{n}$), γ is the surface tension coefficient and k is the mean curvature of the free surface. When the lubrication approximation holds we have

$$k \simeq -\Delta h. \tag{6}$$

After computing the electric field for a circular flat drop and inserting it into the previous equations we arrive at the following lubrication equation:

$$h_t + \frac{1}{r}\left[r\frac{1}{3\mu}h^3\left(\frac{Q^2}{\varepsilon_0\left(4\pi a(t)\right)^2}\frac{1}{a^2(t)-r^2} + \gamma\left(h_{rr} + \frac{h_r}{r}\right)\right)_r\right]_r = 0. \tag{7}$$

This equation is defined for $0 \leq r \leq a(t)$. The initial condition is $h(r, t_0) = h_0(r)$ for $r \in [0, a(t_0)]$ and the boundary conditions are

$$h'(0,t) = h'''(0,t) = 0 \text{ (circular symmetry)}, \; h(a(t),t) = 0 \text{ for } t \geq t_0 \tag{8}$$

together with the condition that no mass is lost at the rim of the drop implying that the limit of the term within brackets in (7) is zero at $r = a(t)$.
Selfsimilar solutions are of the form

$$h = \frac{1}{t^{\frac{1}{5}}}H\left(\frac{r}{t^{\frac{1}{10}}}\right) \tag{9}$$

with $H(\rho)$ satisfying:

$$-Z(\rho H_\rho + 2H) + \frac{1}{\rho}\left[\rho H^3\left(\frac{XY}{1-\rho^2} + H_{\rho\rho} + \frac{H_\rho}{\rho}\right)_\rho\right]_\rho = 0, \tag{10}$$

where X, Y and Z are dimensionless parameters. We find an explicit solution

$$H(\rho) = 1 - \rho^2 \tag{11}$$

provided $Z = 2XY, Y = \pi/2$ and broader families of solutions with interesting geometric properties.

References

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