SECOND-ORDER CONE COMPLEMENTARITY FORMULATION FOR QUASI-STATIC INCREMENTAL FRICTIONAL CONTACT PROBLEM IN THREE-DIMENSIONAL SPACE

Y. Kanno*, A. Pinto da Costa**, J.A.C. Martins**

*Department of Architecture and Architectural Engineering, Kyoto University, Kyoto $606-8501$, Japan

**Instituto Superior Técnico, Departamento de Engenharia Civil e Arquitectura and ICIST, Av. Rovisco Pais, $1049-001$ Lisboa, Portugal

Summary The numerical solution of quasi-static incremental frictional contact problems involving discrete versions of two- and three-dimensional elastic solids or structures is sought with a Second Order Cone Linear Complementarity Problem (SOCLCP) formulation. The Coulomb friction cone is considered without any pyramidal approximation. Some illustrative numerical examples are given.

The SOCLP formulation

In this paper we deal with quasi-static problems in which finite-dimensional geometrically linear elastic structures may establish frictional contact with the surface of rigid obstacles. The nonlinearity of the three-dimensional Coulomb friction cone makes impossible the direct use of linear complementarity formulations to deal with three-dimensional frictional contact problems. In order to overcome this difficulty several formulations use pyramidal approximations of the friction cone. In this paper we consider the classical three-dimensional Coulomb friction cone without any pyramidal approximation. For a contact candidate node in the three-dimensional space, $(\Delta\mathbf{u}_{\rm t}, \Delta u_{\rm n}) \in \mathbb{R}^2 \times \mathbb{R}$ and $(\mathbf{r}_{\rm t}, r_{\rm n}) \in \mathbb{R}^2 \times \mathbb{R}$ denote the vector of incremental displacements and the vector of reactions, respectively. Here, the subscripts $\rm t$ and $\rm n$ denote the two tangential and the normal directions to the obstacle surface, respectively. Denoting the coefficient of friction by $\mu > 0$, Coulomb's friction law

$$\mu r_{\rm n} \geq \|\mathbf{r}_{\rm t}\|, \quad \mathbf{r}_{\rm t} \cdot \Delta\mathbf{u}_{\rm t} + \mu r_{\rm n} \|\Delta\mathbf{u}_{\rm t}\| = 0, \tag{1}$$

can be written as the following linear complementarity condition over two second-order cones [2]

$$(\lambda_{\rm n}, \Delta\mathbf{u}_{\rm t}) \cdot (\mu r_{\rm n}, \mathbf{r}_{\rm t}) = 0, \quad \lambda_{\rm n} \geq \|\Delta\mathbf{u}_{\rm t}\|, \quad \mu r_{\rm n} \geq \|\mathbf{r}_{\rm t}\|, \tag{2}$$

where $\lambda_{\rm n} \in \mathbb{R}$. The unilateral contact condition can be written as

$$\Delta u_{\rm n} - g \geq 0, \quad r_{\rm n} \geq 0, \quad (\Delta u_{\rm n} - g) r_{\rm n} = 0, \tag{3}$$

where g denotes the current distance to the obstacle. Let n^c denote the number of contact candidate nodes. The equilibrium equations after a condensation on the contact candidate nodes are

$$\mathbf{K}\Delta\mathbf{u} = \mathbf{r} + \mathbf{f}, \tag{4}$$

where $\mathbf{K} \in \mathbb{R}^{3n^c \times 3n^c}$ is the condensed stiffness matrix, $\Delta\mathbf{u} \in \mathbb{R}^{3n^c}$ and $\mathbf{r} \in \mathbb{R}^{3n^c}$ denote, respectively, the vector of incremental nodal displacements and the reactions at the contact candidate nodes and $\mathbf{f} \in \mathbb{R}^{3n^c}$ denotes the vector of independent terms that combines effects of applied forces, current gap and condensation process. Conditions (2) – (4), are equivalent to the following *second-order cone linear complementarity problem* (*SOCLCP*)

$$\text{find } (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{4n^c} \times \mathbb{R}^{4n^c} \text{such that}$$
$$\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{q}, \quad \mathbf{x} \in \mathcal{K}_S, \quad \mathbf{y} \in \mathcal{K}_S, \quad \mathbf{x}^{\rm T}\mathbf{y} = 0, \tag{5}$$

where $\mathbf{x} = (\lambda_{\rm n}, \Delta\mathbf{u}_{\rm t}, \Delta\mathbf{u}_{\rm n})$, $\mathbf{y} = (\mu\mathbf{r}_{\rm n}, \mathbf{r}_{\rm t}, \mathbf{r}_{\rm n})$, $\mathbf{M}$ and $\mathbf{q}$ are a matrix and a vector of dimension $4n^c$ and the second-order cone $\mathcal{K}_S = \mathcal{K}_1 \times \mathbb{R}^{n^c}_+ \subset \mathbb{R}^{3n^c}$ with $\mathcal{K}_1 = \{(\mathbf{s}_1, \mathbf{s}_2) \in \mathbb{R}^{n^c} \times \mathbb{R}^{2n^c} | s_{1i} \geq \|\mathbf{s}_{2i}\|\}$. Problem (5) can be solved efficiently by using the recently developed algorithm [1] that combines smoothing and regularization procedures, and is based on the Euclidean Jordan algebra on second-order cones.

Numerical examples

In this section we consider two numerical examples solved by the present formulation and algorithm. The first example is a double layer truss with 12×12 contact candidate nodes with an horizontal flat obstacle at $z = 0$ (Figure 1(b)). Figure 1(a) illustrates a 4×4 truss. The lengths of the members in the x- and y-directions are 2000.0 mm and 3000.0 mm, respectively, and the distance between the upper and lower layers is 2000.0 mm. The elastic modulus and the cross-sectional area for each member are 205.8 GPa and $10000.0\ \mathrm{mm}^2$, respectively. The coefficient of friction is $\mu = 0.12$. The external loads applied to the nodes of the upper layer at load step k are $\gamma^{(k)}\hat{\mathbf{f}}$ for each interior node (denoted by ○; see Figure 1 (a)), $\gamma^{(k)}\hat{\mathbf{f}}/2$ for each edge node (denoted by ●) and $\gamma^{(k)}\hat{\mathbf{f}}/4$ for each corner node (denoted by $\oplus$), where $\hat{\mathbf{f}} = (0, 0, -102.9)$ kN. The sequence of the load parameters $\gamma^{(k)}$ is 0, 5.00, 4.05, 3.1, 2.15, 1.20, 0.25 for $k = 0, \ldots, 6$; after an initial compression against the obstacle the downward forces are progressively alleviated. For $k = 6$ Figure 1(b) illustrates the configuration of the truss while Figure 1(c) shows the projection onto the xy-plane of the total displacements of the contact candidate nodes. The pattern of the residual tangential contact displacements is clearly observed in Figure 1(c) .

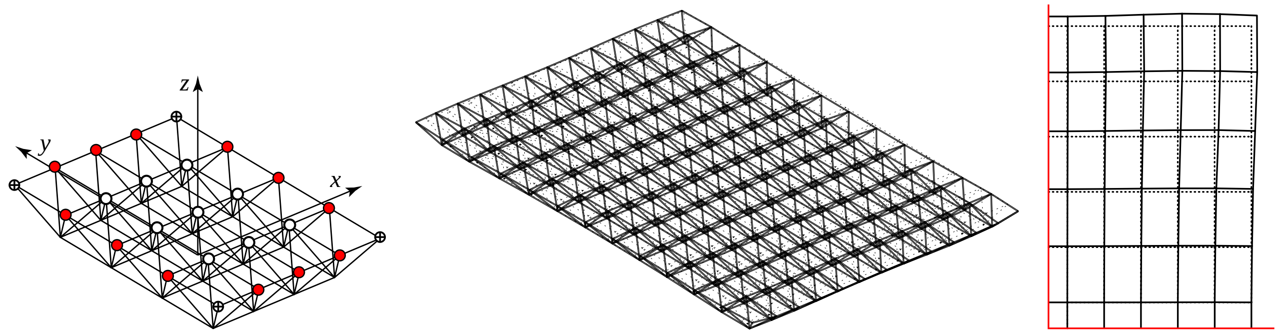

Figure 1. (a) A 4×4 double layer truss. At the end of the loading process (displacements amplified 1000 times): (b) perspective of the equilibrium configuration of the truss; (c) horizontal projection of one fourth of the contact candidate nodes (double symmetry).

The next example is a well known example in the literature. It consists of a linear elastic body (Young's modulus $E = 130$ GPa, Poisson's ratio $\nu = 0.2$) in a state of plane strain. One half of it occupies a 40 mm $\times$ 40 mm square domain as illustrated in Figure 2 and is discretized in 512 bilinear (Q1) finite elements. The right boundary of that square belongs to an axis of symmetry, hence its points are constrained to move only in the vertical direction. The nodes of the bottom segment are submitted to frictional contact conditions ($\mu_i = 1,\ i = 1, \ldots, 33$). The left and top boundary segments are submitted to a monotonic proportional loading that consists of 100 MPa rightward and 50 MPa downward uniform stresses, respectively. The self-weight is neglected. The contact stresses are represented in Figure 2 and match the results presented in previous works; the tangential reaction stresses point to the right.

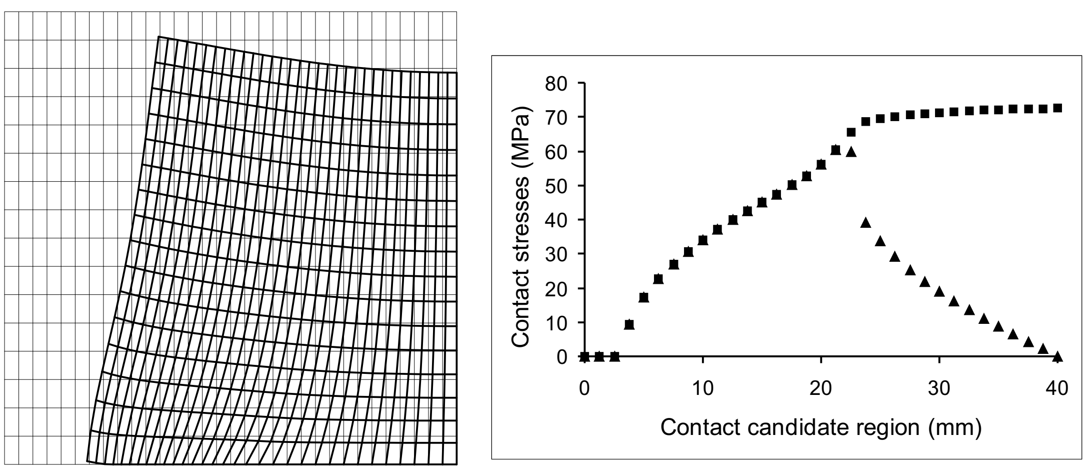

Figure 2. (a) Deformed configuration at an equilibrium state corresponding to a rightward pressure of 100 MPa on the left face and a downward pressure of 50 MPa on the upper face (displacements amplified 500 times); (b) Schematic representation of the normal contact stresses (■) and tangential contact stresses divided by μ (▲).

Conclusions

The equilibrium states of three- and two-dimensional structures and solids were computed by solving the incremental quasi-static frictional contact problem formulated as a SOCLCP. With this formulation no pyramidal approximation of the Coulomb friction cone is needed. The SOCLCP's were solved by a combined smoothing and regularization method.

References

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[2] Kanno Y., Ohsaki M.: Minimum principle of complementary energy of cable networks by using second-order cone programming. *Int. J. Solids Struct.* **40**: 4437–4460, 2003.