T-INCLUSION REGIONS FOR THE EFFECTIVE TRANSPORT COEFFICIENTS OF TWO–PHASE MEDIA

S. Tokarzewski, A. Gałka, B. Gambin, J.J. Telega.
Polish Academy of Sciences IPPT PAN, Swiętokrzyska 21, 00–049 Warsaw, Poland

Summary By starting from power series expanded at a number of real points and infinity we derive the fundamental inclusion relations for the special T-inclusion regions estimating in the complex domain the effective transport coefficients of two-phase media. The T- inclusion regions derived are new and the best over the entire class of rational functions and the given input data. For a particular cases they reduce to the classical complex bounds derived by Bergman [1] and Milton [2], and rederived in a different manner by Tokarzewski and Telega [4]. Nontrivial numerical examples illustrating the results obtained are also provided..

FORMULATION OF THE MATCHING PROBLEM

A macroscopic modelling of microinhomogeneous media and composites requires the evaluation of effective moduli. However their exact values are available only in specific cases; for instance in the one-dimensional periodic homogenization. In the relevant literature many papers deal with estimating of the effective coefficients of two-phase media, such as dielectric constants, magnetic permeabilities, thermal and electrical conductivities.

Assume that the macroscopic response $q_{ef}(z)/q_1$, $z = h - 1$, $h = q_2/q_1$ of two-phase composite is isotropic, where q_1 and q_2 characterize the mechanical properties of two materials the composite is made of. The following problem arises: how to determine the best bounds on $q_{ef}(z)/q_1$ in terms of its power expansions at finite number of real points $x_1, x_2,\ ...,\ x_N < \infty$ and at infinity?

It is well known that the effective transport coefficient of a two-phase medium with macroscopically isotropic symmetry has a Stieltjes integral representation given by

$$f_1(z) = \tfrac{Q(z)-1}{z} = \int_0^1 \tfrac{d\gamma_1(u)}{1+zu},\ d\gamma_1(u) > 0,\ z = h-1,\ h = \tfrac{q_2}{q_1},\ Q(z) = \tfrac{q_{ef}(z)}{q_1};\ f_1(-1) \le 1. \tag{1}$$

Let the truncated power expansions of $f_1(z)$ at $x_1,\ x_2, ...,\ x_N,\ \infty,\ -1$

$$\begin{gathered} f_1(z) = \sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i + O((z-x_j)^{p_j}),\ j = 1,2,...,N;\ \ f_1(z) = \frac{1}{z}\sum_{i=0}^{p_{N+1}-1} c_{i(N+1)}\left(\tfrac{1}{z}\right)^i + O\left(\tfrac{1}{z}\right)^{p_{N+1}} \\ f_1(z) = f_1(-1) + O(z+1);\ f_1(-1) \le 1 \end{gathered} \tag{2}$$

be given, where $-1 < x_j,\ j = 1,2,...,N+1$. We use the following notation

$$f_1(z) = f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}},\ \boldsymbol{x} = [x_1, x_2, ..., \infty, -1]\,,\ \boldsymbol{p} = [p_1, p_2, ..., p_\infty, 1]\,,\ f_1(-1) \le 1. \tag{3}$$

Problem *By starting from the power expansions (3), construct in a complex plane the general T-inclusion regions estimating both the Stieltjes function $f_1(z)$ and via (1_1) the effective coefficient $Q(z)$ of the two-phase composite.*

T-INCLUSION REGIONS

In the first step the inequality $f_1(-1) \le 1$ is replaced by the equality $f_1(-1) = 1$, cf. (3). The input data (3) transform to

$$f_1(z) = f_1|_{x_1,...,\infty\ ,1}^{p_1,...,p_\infty,1}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}}, \quad f_1(-1) = 1. \tag{4}$$

Next by introducing $P_0 = 0,\ P_j = \sum_{i=1}^{j} p_i,\ j = 1,2,...,N;\quad P = 2P_N + 1$ and using linear fractional transformation of type T we expand the given Stieltjes function $f_1(z)$ to the new Stieltjes one $f_{P.}(z)$:

$$\begin{gathered} f_{P_0+1}(z) = \tfrac{f_{P_0+1}(x_1)}{1+(z-x_1)f_{P_0+2}(z)},\ = \tfrac{f_{P_0+1}(x_1)}{1+(z-x_1)G_{P_0+2}+(z-x_1)f_{P_0+3}(z)}, ..., f_{P_1-1}(z) = \tfrac{f_{P_1-1}(x_1)}{1+(z-x_1)G_{P_1}+(z-x_1)f_{P_1+1}(z)}, \\ f_{P_1+1}(z) = \tfrac{f_{P_1+1}(x_2)}{1+(z-x_2)f_{P_1+2}(z)},\ = \tfrac{f_{P_1+1}(x_2)}{1+(z-x_2)G_{P_1+1}+(z-x_2)f_{P_1+2}(z)}, ..., f_{P_2-2}(z) = \tfrac{f_{P_2-1}(x_2)}{1+(z-x_2)G_{P_2}+(z-x_2)f_{P_2+1}(z)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{P_{N-1}+1}(z) = \tfrac{f_{P_{N-1}+1.}(x_{N-1})}{1+(z-x_{N-1})f_{P_{N-1}+2.}(z)},\ \cdots,\ f_{P_N-1}(z) = \tfrac{f_{P_N-1}(x_N)}{1+(z-x_N)f_{P-1.}(z)},\ = \tfrac{f_{P_N-1}(x_N)}{1+(z-x_N)G_{P-1}+(z-x_N)f_{P.}(z)}, \end{gathered} \tag{5}$$

where $G_{2i} = f_{2i}(\infty) > 0$, $i = 1,2,,...,p_\infty$ and 0 otherwise. It is convenient to rewrite (5) as follows

$$f_1(z) = f_1{}_{x_1,...,\infty\ ,1}^{p_1,...,p_\infty,1}(z, f_P(z)),\ \text{where}\ f_P(z) = \int_0^1 \tfrac{d\gamma_P(u)}{1+zu},\ f_P(-1) = w_P. \tag{6}$$

The T-multipoint continued fraction expansion $f_1{}_{\boldsymbol{x}}^{\boldsymbol{p}}(z, f_P(z))$ of $f_1(z)$ matches the input data (4) for any Stieltjes functions $f_P(z) = \int_0^1 \frac{d\gamma_P(u)}{1+zu}$ satisfying $\int_0^1 \frac{d\gamma_P(u)}{1-u} = w_P$. On account of that the T-inclusion region

$\Phi_P(z)$ should incorporate all admissible values of the Stieltjes functions $f_1(z)$ matching the input data (3). Hence we have

$$f_1(z) \in \Phi_P(z) = \{F_P(z,u);\ 0 \le v \le 1,\ -1 \le u \le 1\}\,,$$
$$F_P(z,u) = f_1{}^{p_1,\dots,p_\infty,1}_{x_1,\dots,\infty,1}(z, vw_p F_0(z,u)),\ F_0(z,u) = (1+u \text{ if } -1 \le u \le \text{ and } \tfrac{1-u}{1+zu} \text{ if } 0 \le u \le 1). \tag{7}$$

Here $F_0(z,u)$ is called the elementary bounding function. It consists of two lines, the straight line $1+u$ if $-1 \le u \le 0$ and the arc of the circle $(1-u)/(1+zu)$ if $0 \le u \le 1$, see (7_2).

FUNDAMENTAL RELATIONS FOR T-INCLUSION REGIONS

Now we are prepared to formulate the main result of this contribution reading by the following theorem:.
Theorem 1. *Let the truncated power expansions of Stieltjes function* $f_1(z)$

$$f_1(z) = f_1|_{\boldsymbol{x}}^{\boldsymbol{p}(k)}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}(k)},\quad \boldsymbol{x} = [x_1, x_2, ..., x_N, \infty, -1]\,,\ \boldsymbol{p}(k) = [p_1(k), p_2(k), ..., p_\infty(k), 1] \tag{8}$$

be given, where $k = \sum_{j=1}^{N} p_j + p_\infty + 1$. *Then* T*-inclusion regions* $\Phi_{p(k)}(z)$ *generated by the truncated series (8) satisfy the following fundamental relations*

$$f_1(z) \in \Phi_{P(k)}(z) \subset \Phi_{P(k-1)}(z),\ k = 2, 3, \dots\,,. \tag{9}$$

provided $p(k) \le p(k+1),\ k = 1, 2, ...,$ *i.e.* $p_j(k+1) - p_j(k) \ge 0,\ j = 1, 2, ...$ *These inclusion relations imply that the best* $\Phi_{P(k)}(z)$ *estimate of* $f_1(z)$ *is obtained using only the given number of power series coefficients (*k *is fixed) and that the use of additional coefficients (higher* k*) improves the estimate* $\Phi_{P(k)}(z)$.

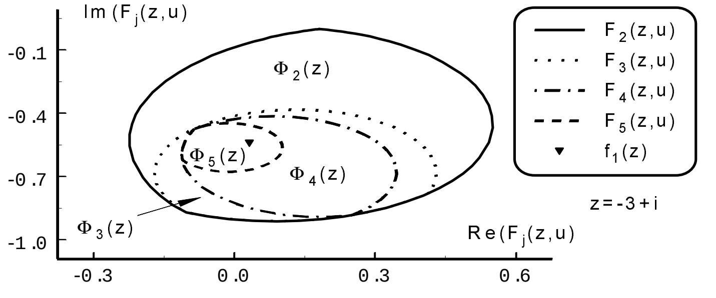

Fig. 1: Monotonic sequence of T-inclusion regions generated by the power series $f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}(k)}$, $\boldsymbol{x}= [0, 3, \infty, -1]$ $\boldsymbol{p}(2) = [1, 0, 0, 1]$, $\boldsymbol{p}(3) = [1, 0, 1, 1]$, $\boldsymbol{p}(4) = [1, 1, 1, 1]$, $\boldsymbol{p}(5) = [1, 1, 2, 1]$ representing the Stieltjes function $f_1(z) = \dfrac{1}{z}\left(1 + \frac{2.5}{z} \ln \frac{1.2+0.1z}{2+0.5z}\right)$, cf. (9)

For $p_\infty = 0$ we have $G_{2i} = 0,\ i = 1, 2, ...,\ p_\infty$, see (3) and (5). For such a case the T-continued fraction expansion of $f_1(z)$ transforms to the S-continued fraction one reported in mathematical literature. Moreover, the substitution $p_1 = 1, p_2 = 1, ..., p_\infty = 0$ reduces the matching problem (3) to the fitting one investigated by Bergman [1] and Milton [2], see also Tokarzewski *et al* [3.4].

FINAL REMARKS

By starting from the several truncated power series we derive, using special T-multipoint continued fraction technique, the general T-inclusion regions estimating in a complex domain the effective transport coefficients $Q(z)$ of two-phase media such as dielectric or diffusion constants, thermal or electrical conductivities, magnetic permeabilities. The T-inclusion regions obtained are new and furnish the best estimates of $Q(z)$. In special cases they reduce to the known complex bounds. Numerical examples exhibiting the usefulness of the results obtained are also provided.

Acknowledgment The authors were also supported by the State Committee for Scientific Research (KBN, Poland) through the Grant No 8 T07A 052 21 and No 3 P04D 007 25

References

[1] D.J. Bergman, Hierarchies of Stieltjes functions and their application to the calculation of bounds for the dielectric constant of a two-components composite medium, SIAM J. Appl. Math., **53**:915–930, 1993.

[2] G.W. Milton, Bounds on the complex permittivity of two-component composite materials, J. Appl. Phys.,**52**:5286–5293, 1981.

[3] S. Tokarzewski, J.J. Telega, S- Continued fraction method for the investigation of a complex dielectric constant of two- components composite, Acta Applicandae Mathematicae, **49**: 55-83, 1997.

[4] S. Tokarzewski, J.J. Telega, M. Pindor, J. Gilewicz, A note on total bounds on complex transport moduli of parametric two-phase media, Z. angew. Math. Phys., **54**: 713-726, 2003.