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# THE STIFFNESS OF PRESTRESSED FRAMEWORKS: A UNIFYING APPROACH

Simon D. Guest
*Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK*

*Summary* Three different methods are commonly used to analyse the stiffness of prestressed frameworks that consist of bars and spherical joints. This paper will show that in fact the correct tangent stiffness matrix for a prestressed structure, as used in the computational literature, can be written as the sum of two terms. The first is a minor modification of a conventional stiffness matrix, that is best understood in terms of the equilibrium matrix that has been widely studied in the engineering literature. The second term is the stress matrix, that has been widely studied in the mathematical rigidity theory literature.

## INTRODUCTION

There are many types of structure where stress stiffening plays a key role in determining structural response. Two obvious engineering examples are tensegrity structures and cable-roof structures; it has also recently been suggested that prestress may have a critical role to play in the stiffness of biological cells [1]. In each of these cases, understanding of the effect of prestress on the stiffness of the structure is critical to understanding how the structure works.

This paper will consider the stiffness of *framework* structures, spherically jointed structures where the structure carries loads by tension or compression of its members, and not in bending. The literature contains three different approaches to calculating the stiffness of these structures; in their own context, each is correct, but it is not clear from the literature how they are related to one another. Making clear the connections between these methods is the aim of this paper.

## PREVIOUS APPROACHES

There are essentially three different methods that have previously been applied to finding the stiffness of prestressed frameworks: each will be described briefly here. I will refer to them as the 'computational' approach, the 'equilibrium' approach, and the 'stress' approach.

Papers describing the 'computational' approach (e.g. [2, 3]) show how to correctly set up the local tangent-stiffness matrix $\mathbf{K}$ for a prestressed framework. Generally these papers describe $\mathbf{K}$ in terms of two components

$$\mathbf{K} = \mathbf{K}_{\text{conv}} + \mathbf{K}_{\text{geom}} \tag{1}$$

where $\mathbf{K}_{\text{conv}}$ is a conventional stiffness matrix of the same geometrically identical, but unstressed, structure, and $\mathbf{K}_{\text{geom}}$ is the change to this stiffness that comes about because of prestress. This approach has proved to be highly successful in calculating the response of various structures. However, the approach itself, as distinct from the results, gives little insight into the behaviour of the structures. The remaining two approaches try to give a more intuitive understanding of the behaviour.

The 'equilibrium' approach (e.g. [4, 5]) does not directly define a stiffness matrix, but instead considers *equilibrium* and *compatibility* relationships, written as an equilibrium matrix $\mathbf{A}$ and a compatibility matrix $\mathbf{A}^{\text{T}}$. This approach allows the calculation, for instance, of all possible states of self-stress and mechanisms of the structure. The response of the structure is described in terms of *extensional* modes, that correspond to the stiffness given by $\mathbf{K}_{\text{conv}}$, and *inextensional* modes that correspond to mechanisms of the unstressed structure; the effect of the states of self-stress on the stiffness of these inextensional modes is then calculated.

The 'stress' approach (e.g. [6, 7]) makes no attempt to find the actual stiffness of a prestressed framework. Rather, the stiffness given by a *stress* matrix $\mathbf{S}$ is explored. The stress matrix has the same form as the stiffness matrix, i.e. it relates displacement and forces, and in general has the following form:

- The submatrix (of size $3 \times 3$ in 3D) of $\mathbf{S}$ relating two nodes $i$ and $j$ that are connected by a member contains $-(t_{ij}/l_{ij})\mathbf{I} = -\hat{t}_{ij}\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, $t_{ij}$ is the tension in the member, $l_{ij}$ is its length, and $\hat{t}_{ij} = (t_{ij}/l_{ij})$ is the *tension coefficient* of the member;
- The diagonal submatrix of $\mathbf{S}$ corresponding to node $i$ contains $\sum_k \hat{t}_{ik}\mathbf{I}$, where the summation is over the nodes that are connected to node $i$ by members of the structure;
- The rest of $\mathbf{S}$ is empty.

Note that the stress matrix takes no account of the elastic properties of the element. This approach is widely used by mathematicians in *rigidity theory*, and has proved highly useful in exploring whether structures will be stiffened by prestress, or not. The stress matrix is highly structured, and this has, for instance, allowed catalogues of symmetric tensegrities to be automatically generated, many of which were previously unknown [8]. However, it has not previously been explored how this approach is related to the other engineering approaches that have been described.

## TANGENT STIFFNESS CALCULATION

The relationship between the different approaches to finding stiffness is most easily illustrated by examining the stiffness of a single bar in a 3-dimensional structure. Consider a bar of length $l$ with axial stiffness $g$ (for an elastic bar $g = AE/l$, where $A$ is the cross-sectional area, and $E$ is the Young's Modulus of the material), that is currently carrying a tension $t$, giving a tension coefficient $\hat{t} = t/l$. A unit vector along the bar from node 1 to 2 is given by the $3 \times 1$ unit vector $\mathbf{n}$. Differentiating the equilibrium relationship at each node allows the tangent stiffness $\mathbf{K}$ to be calculated. For the single bar (denoted by a subscript $s$), this tangent stiffness is given by

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} [g] \begin{bmatrix} \mathbf{n}^{\mathrm{T}} & -\mathbf{n}^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} [\hat{t}] \begin{bmatrix} \mathbf{n} & -\mathbf{n} \end{bmatrix} + \begin{bmatrix} \hat{t}\mathbf{I}_3 & -\hat{t}\mathbf{I}_3 \\ -\hat{t}\mathbf{I}_3 & \hat{t}\mathbf{I}_3 \end{bmatrix} \tag{2}$$

where $\mathbf{I}_3$ is a $3 \times 3$ identity matrix.

Equation 2 can be rewritten using the equilibrium matrix and the stress matrix. For the single bar, the equilibrium matrix is $\mathbf{A}_s = \begin{bmatrix} \mathbf{n}^{\mathrm{T}} & -\mathbf{n}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and the compatibility matrix is $\mathbf{A}_s^{\mathrm{T}}$. The final term in (2) is, for a single bar, exactly the stress matrix described earlier, $\mathbf{S}_s$. Thus,

$$\mathbf{K}_s = \mathbf{A}_s [g] \mathbf{A}_s^{\mathrm{T}} - \mathbf{A}_s [\hat{t}] \mathbf{A}_s^{\mathrm{T}} + \mathbf{S}_s \tag{3}$$

For a structure made up of multiple bars, standard summation techniques allow the assembly of the complete stiffness matrix, which can be written, following the form of (3) as

$$\mathbf{K} = \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\hat{\mathbf{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{S} \tag{4}$$

where $\mathbf{G}$ is a diagonal matrix of the axial stiffness of members and $\hat{\mathbf{T}}$ is a diagonal matrix of the tension coefficients of the members.

Comparing (1) and (4), the conventional stiffness and geometrical stiffness matrices can be written as $\mathbf{K}_{\text{conv}} = \mathbf{A}\mathbf{G}\mathbf{A}^{\mathrm{T}}$, and $\mathbf{K}_{\text{geom}} = -\mathbf{A}\hat{\mathbf{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{S}$, but it is also clear that doing so does not make best use of the structure inherent in these matrices. Rather, it is better to write

$$\mathbf{K} = \mathbf{A}\mathbf{G}'\mathbf{A}^{\mathrm{T}} + \mathbf{S} \tag{5}$$

where $\mathbf{G}' = \mathbf{G} - \hat{\mathbf{T}}$ is a diagonal matrix of *modified* axial stiffness of the members, where each entry is $g' = g - \hat{t}$. For practical structures $\hat{t} \ll g$, and the stiffness given by $\mathbf{K}_{\text{mod}} = \mathbf{A}\mathbf{G}'\mathbf{A}^{\mathrm{T}}$ will be little different from the stiffness $\mathbf{K}_{\text{conv}}$. More importantly, the interesting mechanics associated with $\mathbf{K}_{\text{mod}}$, e.g. zero stiffness modes, are best understood in terms of the equilibrium matrix $\mathbf{A}$.

Thus (5) shows that the correct tangent stiffness, as given by the 'computational' approach, can be written as two terms. The first of these terms is best understood using the 'equilibrium' approach, while the second is exactly the stress matrix used in the 'stress' approach.

## CONCLUSIONS

The paper has shown that the correct tangent stiffness matrix for a prestressed structure is best understood in terms of two terms. The first is a minor modification of a conventional stiffness matrix, that is best understood in terms of the equilibrium matrix that has been widely studied in the engineering literature. The second term is the stress matrix, that has been widely studied in the mathematical rigidity theory literature.

### References


[1] Ingber D.E.: Tensegrity I. Cell structure and hierarchical systems biology. *Journal of Cell Science* **116**(7): 1157–1173, 2003.

[2] Argyris J.H., Scharpf, D.W.: Large deflection analysis of prestressed networks. *Journal of the Structural Division, Proceedings of the American Society of Civil Engineers* **98**(ST3): 633–654, 1973.

[3] Kebiche K., Kazi-Aoual M.N., Motro R.: Geometrical non-linear analysis of tensegrity systems. *Engineering Structures* **21**: 864–676, 1999.

[4] Pellegrino S., Calladine C.R.: Matrix analysis of statically and kinematically indeterminate frameworks. *International Journal of Solids and Structures* **22**(4): 409–428, 1986.

[5] Pellegrino S.: Analysis of prestressed mechanisms. *International Journal of Solids and Structures* **26**(12): 1329–1350, 1990.

[6] Connelly R.: Rigidity and Energy. *Inventiones Mathematicae* **66**: 11–33, 1982.

[7] Connelly R., Terrell, M.: Globally rigid symmetric tensegrities. *Structural Topology* **21**: 59–78, 1995.

[8] Connelly R., Back A.: Mathematics and tensegrity. *American Scientist* **86**: 142–151, 1998.