ON THE CONCEPT OF "DYNAMIC (IN)STABILITY OF QUASI-STATIC PATHS"

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Summary In this paper, a concept of "dynamic stability of quasi-static paths" is discussed that takes into account the existence of fast (dynamic) and slow (quasi-static) time scales. It is essentially a continuity property with respect to the smallness of the initial perturbations (as in Lyapunov stability) and to the smallness of the quasi-static loading rate (that plays the role of the small parameter in singular perturbation problems). Three mechanical examples are presented that illustrate the similarities, the differences and the relations between this concept of "dynamic stability of quasi-static paths" and the one of Lyapunov stability of some related equilibrium configurations or dynamic trajectories.

INTRODUCTION

Force equals mass times acceleration is the classical balance law that governs the motion of mechanical systems. Yet a common and useful approximation for the equations that govern the slow evolution of mechanical systems is to neglect inertia effects and take the balance equations simply as force equals zero. Slow evolutions calculated with the above approximation are called quasi-static evolutions.

The relationship of this issue with the theory of singular perturbations has been established in [3], where the existence of fast (dynamic) and slow (quasi-static) time scales is recognized: a change of variables is performed that replaces the (fast) physical time t by a (slow) loading parameter λ whose rate of change with respect to time, $\varepsilon = d\lambda/dt$, is eventually decreased to zero. This change of variables, done preferably after appropriate non-dimensionalizations are performed, leads to a system of dynamic (ordinary or partial) differential equations that defines (in finite or infinite dimensions) a singular perturbation problem, i.e. a problem that is governed by a system of equations where at least the highest order derivative with respect to λ appears multiplied by a small parameter ε.

An issue that is relevant in the study of quasi-static trajectories is their "stability". The concept of Lyapunov stability has been used for long to study the stability of dynamic trajectories of mechanical systems, namely the zero acceleration trajectories of the equilibrium configurations under constant applied loads. But the application of that concept to quasi-static paths with slowly varying loads faces the difficulty that such paths are not, in general, true dynamic solutions [3]: the supposedly negligible quasi-static accelerations are not null in general.

Energetic or thermodynamic considerations have been used to propose stability criteria for quasi-static paths [1, 4, 5] but such energetic character restricts their application to the case of symmetric stiffness operators and, also, their relationship with dynamics is not always clear. Linearization techniques have also been used to discuss stability of quasi-static paths, but in some cases the non-autonomous character of the linearized equations has been neglected (freezed) [2] and/or it has not been recognized that the quasi-static paths in general do not solve the dynamic equations [4].

The definition of "dynamic stability of a quasi-static path" that we propose in this paper takes inertia into account, recognizes the distinction between the quasi-static and the dynamic governing equations and time scales, and selects the quasi-static paths close to which the dynamic evolutions will remain when:

- the dynamic evolutions initiate sufficiently close to the quasi-static path,
- and the load is applied sufficiently slowly.

DYNAMIC STABILITY OF A QUASI-STATIC PATH AT SOME EQUILIBRIUM POINT

The proposed mathematical definition of dynamic stability of a quasi-static path at an equilibrium state is presented in the context of the following finite-dimensional (dynamic) system of differential inclusions and initial conditions:

$$\begin{cases} \varepsilon\,\mathbf{y}'(\lambda) \in \psi(\mathbf{x}(\lambda), \mathbf{y}(\lambda), \lambda, \varepsilon) \\ \mathbf{x}'(\lambda) \in \phi(\mathbf{x}(\lambda), \mathbf{y}(\lambda), \lambda, \varepsilon) \end{cases} \qquad \begin{Bmatrix} \mathbf{y}(\lambda) \\ \mathbf{x}(\lambda) \end{Bmatrix} \in \mathcal{D} \qquad \begin{Bmatrix} \mathbf{y}(\lambda_0) \\ \mathbf{x}(\lambda_0) \end{Bmatrix} = \begin{Bmatrix} \mathbf{y}_0 \\ \mathbf{x}_0 \end{Bmatrix} \in \mathcal{D}. \qquad (1)$$

Here $\mathcal{D}$ is the domain of admissibility for (some of) the variables in presence, the notation $(\ )' = d(\ )/d\lambda$ is adopted, and, recalling that $\varepsilon = d\lambda/dt$, the derivatives with respect to the physical time t are related to the derivatives with respect to the load parameter λ by $d(\ )/dt = \varepsilon\, d(\ )/d\lambda = \varepsilon\,(\ )'$. In a (finite-dimensional, rate independent) elastic-plastic problem (like the Shanley column [6]), the vector $\mathbf{y}$ is composed of the N generalized displacements of the system, $\mathbf{u}$, plus their corresponding time rates of change, $\mathbf{v}$ (velocities), while the vector $\mathbf{x}$ contains the internal variables α of the system, namely the generalized plastic strains. In that case, the relevant restriction in the domain $\mathcal{D}$ corresponds to requiring that the generalized forces $\mathbf{A}(\mathbf{u}, \alpha)$ associated to the internal variables α, which depend on $\mathbf{u}$ and α, remain in some convex set C. In that case, ψ is not multivalued, and the $\mathbf{y}'$ differential inclusion in (1) is actually a system of $2N$ first order differential equations, the first N of which are the kinematic relations between the displacements $\mathbf{u}$ and the velocities $\mathbf{v}$, while the second N are the governing equations of Dynamics; on the other hand, ϕ may be multivalued and, actually, it is the normal cone to the set C at the generalized forces $\mathbf{A}(\mathbf{u}, \alpha)$, $N_C[\mathbf{A}(\mathbf{u}, \alpha)]$, so that the $\mathbf{x}'$ differential inclusion is nothing but the flow rule.

$$\mathbf{y}(\lambda) = \begin{Bmatrix} \mathbf{u}(\lambda) \\ \mathbf{v}(\lambda) \end{Bmatrix}, \quad \psi\left(\alpha, \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}, \lambda, \varepsilon\right) = \begin{Bmatrix} \mathbf{v} \\ \mathbf{m}^{-1}[\mathbf{f}^{ext}(\mathbf{u}, \lambda) + \mathbf{f}^{int}(\mathbf{u}, \alpha)] \end{Bmatrix}, \quad \phi\left(\alpha, \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}, \lambda, \varepsilon\right) = N_C[\mathbf{A}(\mathbf{u}, \alpha)], \tag{2}$$

where $\mathbf{m}$ is the mass matrix of the system, and $\mathbf{f}^{ext}(\mathbf{u}, \lambda)$ and $\mathbf{f}^{int}(\mathbf{u}, \alpha)$ denote the vectors of generalized external and internal forces, respectively. The corresponding reduced (quasi-static) system is obtained by letting $\varepsilon = 0$ in (1),

$$\mathbf{0} \in \psi(\mathbf{x}^0(\lambda), \mathbf{y}^0(\lambda), \lambda, 0), \qquad \mathbf{x}^{0\prime}(\lambda) \in \phi(\mathbf{x}^0(\lambda), \mathbf{y}^0(\lambda), \lambda, 0), \tag{3}$$

or, in the quasi-static elastic-plastic case, the rate of change of the displacements with respect to the physical time can be considered null, $\mathbf{v}^0(\lambda) = \mathbf{0}$, while the equilibrium equations and the flow rule are:

$$\mathbf{f}^{ext}(\mathbf{u}^0(\lambda), \lambda) + \mathbf{f}^{int}(\mathbf{u}^0(\lambda), \alpha^0(\lambda)) = \mathbf{0}, \qquad (\alpha^0)'(\lambda) \in N_C[\mathbf{A}(\mathbf{u}^0(\lambda), \alpha^0(\lambda))]. \tag{4}$$

In this context, we say that *the quasi-static path* $(\mathbf{y}^0(\lambda), \mathbf{x}^0(\lambda))$, $\lambda \in I = [\lambda_0, \lambda_1[$ that solves the quasi-static system (3) in I *is dynamically stable at the equilibrium state* $(\mathbf{y}^0(\lambda_0), \mathbf{x}^0(\lambda_0))$ *if there exists* $0 < \Delta\lambda < \lambda_1 - \lambda_0$, *such that, for all* $\delta > 0$, *there exist* $\bar{\rho} > 0$ *and* $\bar{\varepsilon} > 0$ *such that for all initial conditions* $(\mathbf{y}_0, \mathbf{x}_0) \in \mathcal{D}$, *all parameters* $\varepsilon > 0$ *and all load parameters* λ *such that* $\|\mathbf{y}_0 - \mathbf{y}^0(\lambda_0)\| + \|\mathbf{x}_0 - \mathbf{x}^0(\lambda_0)\| < \bar{\rho}$, $\varepsilon < \bar{\varepsilon}$, *and* $\lambda \in [\lambda_0, \lambda_0 + \Delta\lambda]$, *the solutions* $(\mathbf{y}(\lambda), \mathbf{x}(\lambda))$ *of the dynamic problem* (1) *with the value* ε *of the perturbation parameter and the initial conditions in* (1) *satisfy* $\|\mathbf{y}(\lambda) - \mathbf{y}^0(\lambda)\| + \|\mathbf{x}(\lambda) - \mathbf{x}^0(\lambda)\| < \delta$.

This definition is essentially a continuity property of the solutions of (1) with respect to *both* the size of the initial perturbations (as in Lyapunov stability theory) and the value of the small parameter ε (as in singular perturbation theory). In this context we remark that, contrary to the singular perturbation theory, the reduction of ε to zero, *alone*, does not always guarantee the convergence of the dynamic solution to the quasi-static one, even in the case of stable quasi-static paths: the reduction of the initial perturbations may be also needed, namely, in the case of undamped mechanical systems. It may look strange that (unlike the definition of Lyapunov stability) a finite interval of the independent variable λ is considered in the proposed definition, and also that the defined property is termed stability. But it is important to observe that the qualitative behavior of the dynamic evolutions in the neighborhood of different portions of the same quasi-static path may be quite distinct. And it should be kept in mind that finite intervals of the (slow) load parameter correspond to infinitely large intervals of the (fast) physical time t, as the loading rate $\varepsilon = d\lambda/dt$ becomes negligibly small.

THREE EXAMPLES OF DYNAMIC (IN)STABILITY OF QUASI-STATIC PATHS

Three simple mechanical examples are used to illustrate and interpret the differences and the relations between this concept of "dynamic stability of quasi-static paths" and the one of Lyapunov stability. Indeed, there may exist:

- quasi-static trajectories that (in the vanishing load rate limit) may be considered as unstable, and that have all their equilibrium points also unstable (for constant loads); the well-known *Ziegler column* [7] illustrates this situation: not surprisingly, the resulting flutter oscillations for slowly varying loads have amplitudes that vary with the load parameter in the same manner as the amplitudes of the corresponding limit cycles obtained at constant load.
- quasi-static trajectories that are made of stable equilibrium points (for constant loads) but that may be considered as unstable (in the vanishing load rate limit); the well-known *Shanley column* [6] illustrates this situation: the dynamic solutions initiating close to the stable (for constant load) equilibrium positions of a portion of the fundamental path from which quasi-static solutions bifurcate with increasing load, do not necessarily remain close to that fundamental quasi-static path.
- and quasi-static trajectories that (in the vanishing load rate limit) may be considered stable, although they coincide with dynamic trajectories that are Lyapunov unstable for any non-vanishing load rate; a *pin-on-flat friction apparatus* discussed here has some features in common with a more complex problem [3] in non-associative elasto-plasticity: the amplitude of the limit cycle oscillations due to the expected flutter instability depends on the load rate ε and becomes vanishingly small as ε is decreased to zero.

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