DISCONTINUOUS TRANSFORMATIONS AND AVERAGING FOR VIBRO-IMPACT ANALYSIS

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Summary Certain vibro-impact problems can be conveniently solved by discontinuous transformations combined with averaging. We outline how this can be accomplished by examples: A self-excited friction oscillator with stop(s), and a particle on a vibrating plane.

INTRODUCTION

Vibro-impact systems are characterized by repeated collisions. Applications include devices to crush, grind, forge, drill, punch, tamp, pile, and cut a variety of objects, and vibrating machinery or structures with slips and stops [1]. The classical approach for solving problems in this area is "stitching", i.e. integrating motions between impacts, and using kinematic impact conditions to switch solution intervals. For numerical simulation this is simple and effective. But for obtaining analytical solutions the method is elaborate, and possible only for the most simple systems [2] – and any additional non-linearity makes it very difficult to apply. For typical applications it is not necessary to obtain solutions at the level of detail provided by exact methods; of more interest may be condensed measures such as oscillation frequencies, stationary amplitudes, and the stability of motions. Thus approximate methods are both necessary and useful. Among these are the methods of *equivalent linearisation* [1], *direct partition of motion* [3], and *averaging* [4,5,6], each with their particular strengths. We demonstrate how special non-smooth transformations and non-standard averaging can be conveniently employed. Compared to classical "stitching", this approach works even in the presence of additional nonlinearities, and provides analytical solutions free of switching conditions. By contrast to equivalent linearisation, it assumes a kinematic rather than kinetic impact formulation. Compared to the averaging approach described in [6] the non-smooth functions need not to eliminate the impacts completely, and are thus much easier to set up.

AVERAGING FOR VIBRO-IMPACT SYSTEMS

Standard averaging applies to systems $d\mathbf{x}/d\varphi = \varepsilon\mathbf{f}(\varphi,\mathbf{x})$, $\varepsilon<<1$, $\mathbf{x}(\varphi)\in D\subset R^n$. According to the averaging theorem [7], if $\mathbf{f}$ is 2π-periodic in φ and bounded and Lipschitz-continuous in $\mathbf{x}$ on D, then $\mathbf{x}$ is asymptotically close to the solution $\mathbf{x}_1$ of the averaged system $d\mathbf{x}_1/d\varphi = \varepsilon\langle\mathbf{f}(\varphi,\mathbf{x}_1)\rangle$ on the scale $\varphi = O(1/\varepsilon)$, where $\langle\ \rangle$ denotes averaging over $\varphi\in[0;2\pi]$. With vibro-impact problems formulated with kinematic impact conditions, the inherent discontinuities in velocity precludes using standard averaging. However, a special form of the averaging theorem has been proven [5], that holds for systems with *small* (i.e. $O(\varepsilon)$) discontinuities in the state variables:

$$\text{(a)}\ \frac{d\mathbf{x}}{d\varphi} = \varepsilon\mathbf{f}(\varphi,\mathbf{x})\ \text{ for }\ \varphi\neq j\pi,\ j=0,1,\cdots;\qquad \text{(b)}\ \mathbf{x}_+ - \mathbf{x}_- = \varepsilon\mathbf{g}(\mathbf{x}_-)\ \text{ for }\ \varphi = j\pi \tag{1}$$

where $\mathbf{f}$ satisfies the same requirements as stated above, and $\mathbf{x}_-$ and $\mathbf{x}_+$ are the states $\mathbf{x}$ immediately before and after the passage of φ through the value $j\pi$. For that case the averaged system becomes:

$$\frac{d\mathbf{x}_1}{d\varphi} = \varepsilon\left(\langle\mathbf{f}(\varphi,\mathbf{x}_1)\rangle + \pi^{-1}\mathbf{g}(\mathbf{x}_1)\right) + \mathrm{O}(\varepsilon^2) \tag{2}$$

To transform vibro-impact problems into the form (1) may not be trivial, not least since for near-elastic vibro-impact the discontinuity in velocity is not small. At least two transforms is required: One for transforming large discontinuities into small ones, and another for transforming the impact-free part of the equations of motion into the form (1)a. Below we outline how this can be accomplished for specific cases.

EXAMPLES

Self-excited friction oscillator with a one-sided stop

Fig. 1 shows the classical "mass on moving belt" model [8], though, with a stop at the right restricting motions to $s<\Delta$ (the left stop is ignored for now). Without stop(s) this system is classical for illustrating friction-induced oscillations; for example, [9] uses averaging to derive stationary amplitudes for pure slip and stick-slip oscillations. With one- or two-sided stops, it models rubbing objects with slipping parts, e.g., loosely mounted brake pads. A typical system is:

$$\begin{aligned} &\ddot{s} + s = -\varepsilon h(\dot{s})\quad \text{for } s<\Delta,\quad h(\dot{s}) = h_1\dot{s} + h_2\dot{s}^2 + h_3\dot{s}^3,\ \ \dot{s}<v_b \\ &s_+ = s_-,\ \ \dot{s}_+ = -R\dot{s}_-\quad \text{for } s=\Delta \end{aligned} \tag{3}$$

where $h_1 = 2\beta - k_1 + 3k_3v_b^2$, $h_2 = -3k_3v_b$, $h_3 = k_3$, the friction law is $\mu(v_{rel}) = \mu_s\mathrm{sgn}(v_{rel}) - k_1v_{rel} + k_3v_{rel}^3$, R is the coefficient of restitution, and β the viscous damping coefficient. Our main interest here is in the effect of impacts, so $\dot{s}<v_b$ is assumed to avoid unnecessary complications connected with sticking motions ([9] considers stick-slip).

To express (3) in the form (1) we first apply a discontinuous transformation of the dependent variable: $s = \Delta - |z|$, where the new variable $z(t)$ changes sign at every impact, i.e. $z_+z_-<0$. In the z-variable every other oscillation of s and $\dot{s}$ will be mirrored, so that if $R = 1$ the velocity-discontinuity at impact is eliminated, while for near-elastic impacts the discontinuity will be small (Fig. 2). Specifically the impact condition becomes $z_+ - z_- = -(R-1)z_-$, which is small for $(R-1)<<1$, while the impact-free equation in (3) becomes $\ddot{z} + z = \Delta\,\mathrm{sgn}\,z - h_1\dot{z} + h_2\dot{z}^2\,\mathrm{sgn}\,z - h_3\dot{z}^3$ for $\dot{z} \neq 0$.
Next we employ a Van der Pol transformation $z = A\sin\varphi$, $\dot{z} = A\cos\varphi$, where $A = A(t)$ and $\varphi = t + \psi(t)$. Assuming $\Delta,(1-R), h_1,h_2,h_3 = O(\varepsilon)<<1$, this yields a system of the form (1). Then it is straightforward to employ (2) for calculating the averaged system, and from that derive stationary values of oscillation amplitude $A_{1\infty}$ and frequency $\omega_{1\infty} = \dot{\varphi}_{1\infty} = 1 + \dot{\psi}_{1\infty}$. The resulting expressions are very simple, and provide good agreement with numerical simulation for even quite large values of $R-1$ (but a small Δ is required). In fact most of the (non-sticking) analytical results derived in [9] for the system without stop also hold for the system with one-sided stop – provided that β is replaced by an effective viscous damping $\beta_{eff} = \beta + (1-R)/\pi$, and the low-speed slope h_1 of the friction curve is replaced by $h_{1eff} = 2\beta_{eff} - k_1 + 3k_3v_b^2$. For example, periodic motions with stationary amplitude $A_{1\infty} = (-4h_{1eff}/(3h_3))^{1/2}$ exist and are stable for $h_{1eff} < 0$, where the zero-solution is unstable. The frequency of stationary oscillations becomes $\omega_{1\infty} = 1 - 2\Delta/(\pi A_{1\infty}) - 2h_2A_{1\infty}/(3\pi)$.

... and with a two-sided stop

With both stops (Fig. 1), the impact event in (3) changes to $|s| = \Delta$, and the first equation holds for $|s|<\Delta$. This changes the character of solutions, so that the mirror transformation used with one stop will not reduce the discontinuity in velocity to $O(\varepsilon)$. Instead we employ the transformation $s = 2\Delta\pi^{-1}\Pi(z), \Pi(z) = \int_0^z M(\zeta)d\zeta, M(z) = \mathrm{sgn}\cos z$ (suggested in [5] for $R = 1$; here we use it for $R\approx1$), which unfolds the sawtooth-like solution s to a polygon-like curve with small discontinuities in velocity (Fig. 3). Using z as a new *in*dependent variable, and the total energy $E(z)$ as the new dependent variable, $E = \frac{1}{2}(\dot{z}^2 + Q(z))$, $Q = \int_0^z M(\zeta)\Pi(\zeta)d\zeta$ – the system attains the form (1), which is applicable for averaging by (2). Note that for motions hitting both stops the oscillation amplitude is fixed, so that with increased energy input only the *frequency* of oscillation can increase. The prediction of this frequency agrees quite well with numerical simulation for parameter values such as, e.g., $R = 0.95$ and $\Delta = 0.5$ (for the two-stop case Δ is *not* assumed to be small).

Particle on a vibrating plane in gravity

Relative motions $y(\tau)$ of a particle bouncing on a vertically vibrating horizontal table are governed by $\ddot{y} = -\gamma + a\sin\tau$ for $y>0$ and $\dot{y}_+ = -R\dot{y}_-$ for $y = 0$, where $\tau = \Omega t$, $\Omega>>g/l$, $\gamma = g/l/\Omega^2<<1$ is the gravity to table acceleration ratio, $a<<1$ the table amplitude, and l a characteristic plate dimension. We first transform $y = |z|$, $z_+z_-<0$, to reduce the discontinuity in velocity, and then apply a Van der Pol like transformation $z(\tau) = A(\tau)N(\varphi)$, $\varphi(\tau) = \omega\tau + \psi(\tau)$ to obtain a system of the form (1). Here N defines the normalized exact solution for a particle bouncing with $R = 1$ on a table at rest, i.e. N is expressed as a second-order polynomial in φ (mod 2π), or as the integral of the sawtooth function $dN/d\varphi = 8\pi^{-2}\arcsin\cos\varphi$. Stationary values of (A,ω,ψ) has been determined as the equilibriums for the averaged system (2). The extension of this procedure to vibro-impact on a vibrating elastic plates or membrane is currently investigated.

CONCLUSION

An extended form of averaging can be used for vibro-impact problems with nearly elastic collisions. This can be accomplished by discontinuous transformations, which converts large discontinuities in state variables into small ones.

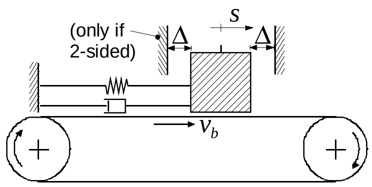

Fig. 1 Self-excited friction-oscillator with stop(s)

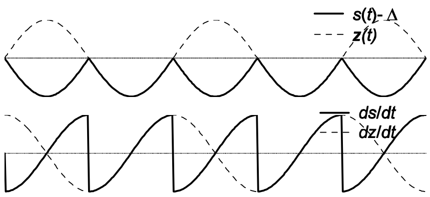

Fig. 2 Motions of friction-oscillator with one-sided stop

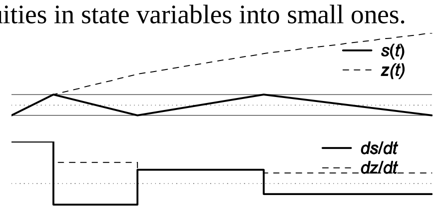

Fig. 3 ... and with a two-sided stop

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