OPTIMIZATION OF ACTIVE CONTROL OF STRUCTURAL VIBRATION BY THE BEAM ANALOGY

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Summary: Optimal control of elastic vibrations by a set of actuators is determined using the finite element (FE) approach. The method combines two FE models, one standard model to represent the structure's dynamics and the second model consisting of fictitious static beams to solve the optimality equations for the problem.

The Problem and the Solution Methodology

Consider an elastic structure under the action of a set of discrete actuators. The structure is governed by the usual equation of motion in the form:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t) \tag{1}$$

where M, C, and K are the symmetric mass, Rayleigh damping, and stiffness matrices respectively. It is assumed that a sufficiently accurate FE model with m degrees of freedom (DOF) is available to simulate the structure's dynamics. The actuators generate the control force vector $F_a(t)$ with the number of independent components equal to n_a, the number of actuators. The vector of nodal forces in equation (1) is $F = BF_a$ where the known placement matrix B has dimension $m \times n_a$. In general, the actuators are to move the structure in time t_f from an initial configuration $x(0) = x_0$, to a final configuration $x(t_f) = x_f$. The manoeuvre time t_f can be given, or may be treated as a variable of optimization. The objective is to determine the action of actuations that will minimize the performance index defined as:

$$J = 1/2\int_0^{t_f}[x^T Q_d x + \dot{x}^T Q_v \dot{x} + F^T RF + G]dt \rightarrow \min \tag{2}$$

where Q_d, Q_v, and R are symmetric positive definite weighting matrices, and Γ is a positive constant.

The above problem is usually handled by applying Pontryagin's Principle (PP) or by the parametric optimization approach. Both methodologies are computationally very intensive [1]. Typically, the PP's application leads to Riccati's coupled non-linear matrix equations the solution of which is *'the most time-consuming part of any optimal control problem'* ([1], p. 109), and the existing procedures '*do not universally guarantee stable and accurate computation*' ([2], p. 248).

A different method, which appears to be numerically efficient, is presented here. For this particular problem the following optimality equations can be derived from the PP[3]:

$$MRM\ddddot{x} + (2KRM - Q_v - CRC)\ddot{x} + (KRK + Q_d)x = 0 \tag{3}$$

The initial conditions, x_0 and $\dot{x}_0$, represent the disturbed, and the final conditions $x_f = \dot{x}_f = 0$ the vibration-free structure. Equation (3) with these conditions constitutes the boundary value (BV) problem in the time domain that can formally be solved by using the FE methodology. This BV problem simplifies significantly if the weighing matrices are assumed to be linear combinations of the matrices M, C, and K, i.e.: $Q_d = a_1M + a_2K + a_3C$, $Q_v = b_1M + b_2K + b_3C$, and $R = [g_1M + g_2K + g_3C]^{-1}$ where a_i, b_i, and g_i are non-negative *optimization constants* (Q_d, Q_v and R will always be positive definite). Next, substituting $x = Fh$, where F is the matrix of M-normalized eigenmodes and $h(t)$ is the vector of modal variables, the optimality equation (3) is transformed into the decoupled equations in the form:

$$\hat{R}_{ii}\ddot{h}_i + [2w_i^2(1-2x_i^2)\hat{R}_{ii} - \hat{Q}_{iiv}]\ddot{h}_i + (w_i^4\hat{R}_{ii} + \hat{Q}_{iid})h_i = 0\,, \qquad i=1...s \tag{4}$$

where w_i are ordered eigenfrequencies (i.e. $0 \le w_1 \le w_2 ... \le w_s$) and x_i are modal damping ratios of the structure. The coefficients in equation (4) are known functions of the optimization parameters and w_i (for example, $\hat{R}_{ii}^{-1} = g_1 + g_2w_i^2 + 2g_3w_ix_i$). Equation (4) can be solved mode by mode for a certain number of s

initial modes (typically $s<<m$). Once it is solved, the optimal control force vector is determined from $F_a = (B^T\Phi\Phi^T B)^{-1}B^T\Phi U$, where s components of vector U are defined by $U_i = \ddot{h}_i + 2x_i w_i \dot{h}_i + w_i^2 h_i$.
A convenient method of solving equations (4) arises from analogy between these equation and the governing equations for a set of s fictitious independent static beam of length L resting on elastic foundation k_{fi}, and loaded by axial compressive force P_i . Each beam is governed by:

$$EI_i v_i'''' + P_i v_i'' + k_{fi} v_i = 0 \qquad i = 1..s \qquad (5)$$

where $(.)' = \frac{\partial}{\partial y}$, and $0 \le y \le L$. Two geometrical boundary conditions can arbitrarily be imposed on each end of the beam. Equations (4) and (5) become analogous if the beam's bending stiffness is selected such that $EI_i \equiv \hat{R}_{ii}$ and if $P_i \equiv 2w_i^2(1-2x_i^2)\hat{R}_{ii} - \hat{Q}_{iiv}$, and $k_{f_i} \equiv w_i^4 \hat{R}_{ii} + \hat{Q}_{iid}$. One can also assume that $L = t_f$ (where L is in meters and t_f is in seconds, for the SI unit system), then the variables of equation (4) relate to the variables of equation (5) as: $t \equiv y$ and $h_i(t) \equiv v_i(y)$. Consequently, for equivalent boundary conditions all the modal functions and their derivatives can be determined in terms of the deflection, slope, and bending moment the fictitious static beams (for example, $U_i = M_i / EI_i + 2w_i x_i q_i + w^2{}_i v_i$).
Besides the already mentioned structural FE model (any structural elements can be used in it), a second FE model, built of fictitious static 2-D beams, is needed to handle equations (5). The latter model can be solved using standard hermitian beam elements. Once the results of this model are known, the optimal actuator forces and the optimal response of the structure can be easily obtained. Note that for verification, the forces calculated from the analogy can be applied to the structural model to determine its 'true' transient dynamic response.
To illustrate the method active vibration control of an elastic aluminum fin modeled by two 2*mm* thick plates as shown in Figure 1 is presented. The dynamic model of the structure has about 2000 DOFs ($m \approx 2000$). If the fin's vibrations are to be attenuated by two actuators (placed at nodes 2 and 12, for example) then two fictitious beams ($s = n_a = 2$) are required to predict the optimal action of these actuators. Either the real disturbances can be considered for an open-loop control, or a somewhat arbitrarily initial configuration can be used to determine optimal gains for a close-loop control.

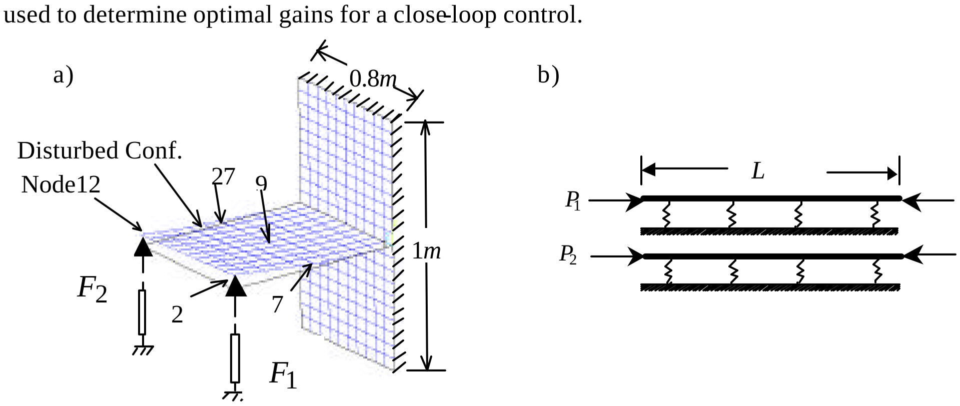

Figure 1. The structure controlled by two actuators (a), and the corresponding fictitious beams (b).

For example, the analysis indicates that the actuators capable of producing about 180*N* are required if $t_f \to \infty$ (for the time-invariant problem), and about 550*N* if the manoeuvre time is limited to 1*s*.

References

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[3] Szyszkowski W, Grewal I S (2000) "*Beam Analogy for Optimal Control of Linear Dynamic Systems*". Computational Mechanics, 25(5): 489-501.