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STABILITY OF PLANE POISEUILLE FLOW AND ENERGY GROWTH IN THE CASE OF A BINGHAM FLUID

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Summary The present work examines the linear stability of three dimensional perturbations of Poiseuille flow of a Bingham fluid. The principal characteristic of the basic flow is the presence of the plug zone which moves as rigid solid. A Chebychev collocation method is applied to compute eigenvalues and the maximum transient amplification factor. It is found that the Poiseuille flow of a Bingham fluid is linearly stable. Due to the non-normality of the operators, a transient amplification of the kinetic energy of disturbances is observed. The results show that the amplification factor decreases with increasing Bingham number. Critical conditions for the onset of energy growth are also determined. When $B >> 1$, it is shown that the critical Reynolds number Re_c increases as $B^{1/2}$.

INTRODUCTION

We consider the flow of an incompressible Bingham fluid with a yield stress τ_y and a plastic viscosity μ_0 in a plane channel. The non-dimensionalized deviatoric extra-stress tensor is given as :

$$\boldsymbol{\tau} = \frac{1}{Re}\left[1 + \frac{B}{D_{II}}\right]\mathbf{D} \quad \Leftrightarrow \quad \tau_{II} > \frac{B}{Re} \quad \text{and} \quad D_{II} = 0 \quad \Leftrightarrow \quad \tau_{II} \leq \frac{B}{Re}, \tag{1}$$

where D_{II} and τ_{II} are respectively the second invariant of the strain rate $\mathbf{D}$ and of the tensor $\boldsymbol{\tau}$. The quantity $(1 + B/D_{II})$ is a dimensionless effective viscosity μ. The dimensionless parameters B and Re are respectively the Bingham and the Reynolds number. They are defined by $B = \tau_y\, H^*/\mu_0\, U_0^*$ and $Re = \rho U_0^*\, H^*/\mu_0$.

The flow in the yielded domain is described by Navier-Stokes equations ; in the region where the yield stress is not exceeded (unyielded region), we have $D_{II} = 0$: this region is called the plug zone ; it moves like a rigid solid. The interface between the viscous region and the plug zone is a yield surface where $\tau_{II} = B/Re$. The motion of the plug region is determined by the conservation of linear momentum. For one dimensional shear flow, we have $\mathbf{U} = (\mathrm{U(y)}, 0, 0)$, with :

$$U(y) = \begin{cases} 1, & 0 \leq |y| < y_0, \\ 1 - \left(\dfrac{|y| - y_0}{1 - y_0}\right)^2, & y_0 \leq |y| \leq 1, \end{cases} \tag{2}$$

where, $\pm y_0$ are the positions of the interfaces. The basic flow depends only on B and is represented in Fig. 1.

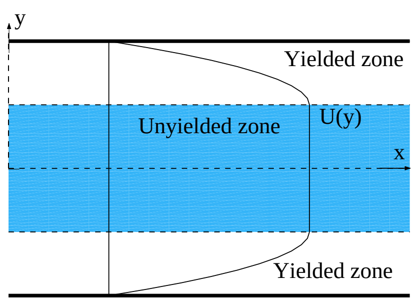

Figure 1. Poiseuille flow of Bingham fluid

LINEAR STABILITY ANALYSIS

Following the usual linear stability analysis, an infinitesimal perturbation $(\epsilon\mathbf{u}', \epsilon\mathrm{p}')$, where $(\epsilon << 1)$ is superimposed on the basic flow $(\mathbf{U}, \mathrm{P})$. Wherever the yield stress is exceeded, the effective viscosity is also perturbed:

$$\mu\left(\mathbf{U} + \epsilon\mathbf{u}'\right) = 1 + \frac{B}{|DU|} - \epsilon\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right).\frac{B}{DU|DU|}, \tag{3}$$

The linearized perturbation equation are then derived. Because the streamwise and spanwise directions are uniform, perturbations are assumed to have the form : $A(x,y,z,t) = \hat{A}(y,t)exp[i(\alpha\, x + \beta\, z)]$, where $\hat{A}$ stands for $\hat{u}, \hat{v}, \hat{w}$ and $\hat{p}$. The resulting equations and the boundary conditions can be found in [2]. These equations can be cast in $\hat{u}, \hat{v}$ formulation if $\beta \neq 0$ or $\hat{v}, \hat{w}$ formulation if $\alpha \neq 0$. They can be written as :

$$\mathcal{L}\,\hat{\mathbf{q}} + \frac{\partial}{\partial t}\mathcal{M}\,\hat{\mathbf{q}} = 0, \tag{4}$$

where $\hat{\mathbf{q}} = (\hat{u}, \hat{v})^T$ or $(\hat{v}, \hat{w})^T$, $\mathcal{L}$ and $\mathcal{M}$ are 2×2 matrices of linear differential operators. An eigenvalue problem is then derived by assuming $\hat{A}(y,t) = \bar{A}(y)exp(-iCt)$. We obtain : $iC\bar{q} = \mathcal{M}^{-1}\,\mathcal{L}\,\bar{q} = \mathcal{L}_1\,\bar{q}$. The eigenvalues are computed, using a spectral collocation method based on Chebychev polynomials. The least stable mode determines the asymptotic behavior of a perturbation initiated at $t = 0$. The analysis of the transient evolution is performed using kinetic energy of the perturbation. Let $G(t)$ the maximum energy amplification at time t, following the same procedure as in [3], $G(t)$ is :

$$G(t) = max\,\frac{\|\mathbf{q}(\mathrm{t})\|^2}{\|\mathbf{q}_0\|^2} = \|exp(-i\,\mathcal{L}_1\,t\|. \tag{5}$$

We have to note that the Squire theorem and the formulation normal velocity and vorticity can not be used here.

RESULTS AND DISCUSSION

The aim of this study is to understand the influence of the Bingham number on the stability problem. For this purpose and by comparison with the Newtonian fluid, one has to note that $B > 0$ induces : (i) a variable effective viscosity in the yielded zone ; (ii) an increase of the viscous dissipation [2] ; (iii) an increase of the velocity gradient near the wall and (iv) a modification of the boundary conditions at the interface.
When $B \to 0$, the plug zone is very thin and the Bingham terms can be neglected in the linearized perturbation equations. The analysis of this particular case denoted after $B \to 0$ permits to assess separately the effect of the boundary conditions at the interface on the stability.
The analysis starts with the eigenvalues spectrum. An example is shown on Fig. 2: it is obtained for $\alpha = 1$, $Re\ (1 - y_0) = 10^4$ and $B\ (1 - y_0) = 2$. The eigenvalues of the P-family are distributed along two branches, in contrast with the Newtonian fluid, for which it consists in one branch. The splitting of the P-family is actually a consequence of the boundary conditions at the interface $u = v = w = 0$. In the case of three dimensional disturbance a splitting of S-family is also observed (Fig 3). This splitting is probably due the coupling between the velocity components when $B > 0$. The calculation shows that for sufficiently large Reynolds numbers, the least stable mode is an interfacial mode. In addition, in the range of parameters considered here, we have not found any instability. We can also add the fact that B has a stabilizing effect. Concerning the transient growth, Fig. 4 gives the amplification factor for a Newtonian fluid and two values of B : $B = 0.001$ and $B = 2$. We observe that even for moderate values of B, G_{max} is strongly reduced by comparison with the Newtonian fluid.
We conclude this section by the determination of the conditions for no energy growth fig. 5. When $B >> 1$, it is shown that the critical Reynolds number Re_c increases as $B^{1/2}$. This asymptotic behavior is in agreement with the conditional stability based on the energy method [1]

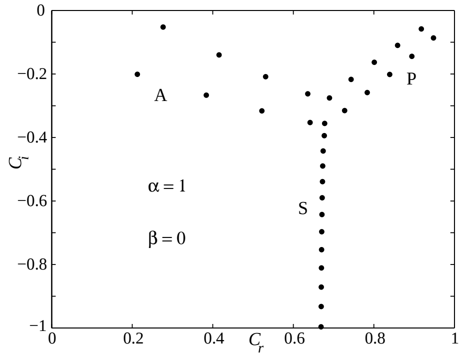

Fig 2 : eigenvalues spectrum for $B \to 0$ and $Re = 10^4$.

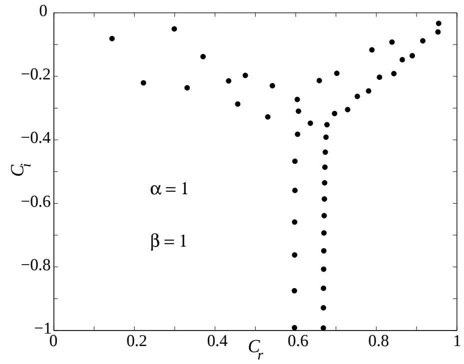

Fig 3 : eigenvalues spectrum for $B = 2$ and $Re = 10^4$.

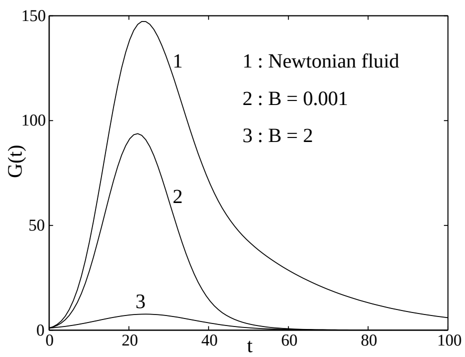

Fig4 : Maximum growth as a function of time.

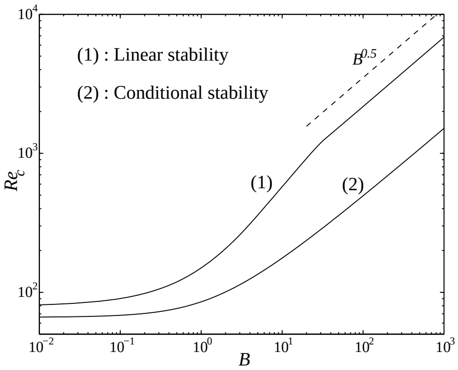

Fig 5 : Critical Reynolds number versus B.

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