BI-DIRECTIONAL WATER WAVES AND INTEGRABLE HIGH ORDER KDV EQUATIONS

Georgy I. Burde*

*J. Blaustein Institute for Desert Research, Ben-Gurion University, 84990 Sede-Boker Campus, Israel

Summary It is shown that the Boussinesq system arising in the context of the shallow water wave motion can be decomposed to a set of coupled equations for the right- and left-moving waves in such a way that, to any order, one of the equations is dependent only on the main right-moving wave and has the form of the KdV equation with higher order corrections. Some classes of exact solutions of the right-moving wave equation, which include impacts of all orders of the asymptotic perturbation expansion, are constructed.

BI-DIRECTIONAL WATER WAVES

The irrotational incompressible flow of shallow layer of inviscid incompressible fluid moving under the influence of gravity as well as surface tension is studied. The wave regime is considered where the surface wave motion has small amplitude and long wavelength and the classical Stokes number is of order one so that the amplitude parameter α and the wavelength parameter β may be treated on an equal footing: Expanding the Euler equations and boundary conditions with respect to the small parameter β and retaining terms up to a certain order gives the Boussinesq equations [1]. These equations admit waves travelling in two directions and thus free of the presumption of unidirectionality that is the hallmark of Korteweg-de Vries (KdV)-type equations as models of surface wave propagation. Although the wave theory first introduced solitons as solutions of unidirectional wave equations, it is expected that the Boussinesq equations should have more intrinsic interest than the unidirectional models.

The KdV equation and its extended versions including higher order corrections are commonly derived from the Boussinesq system by specializing to a wave moving to the right. In the present paper, solutions of the Boussinesq system are considered, in which the surface elevation splits into two components u and ζ corresponding to the right- and left-moving waves (In a similar manner, in [2], the first order system of coupled KdV equations for the right- and left-moving waves is derived and, in [3], the interaction of solitary waves at fourth order is considered under the assumption that the amplitude of the left-moving wave is of $O(\beta^3)$). In the present work, it is assumed that the amplitude of the left-moving wave is of $O(\beta)$ as compared with that of the right-moving wave. The procedure, similar to that commonly used for the unidirectional waves to derive the KdV equation and its extended high order versions, is applied to the Boussinesq equations to decompose them to a set of equations consisting of a relation expressing the fluid velocity through u and ζ and a system of two coupled equations for u and ζ. It is shown that a non-uniqueness of such a decomposition can be used to derive a system of equations for u and ζ, in which, to any order, one of the equations is dependent only on the surface elevation u for the main right-moving wave while the second equation includes both u and ζ. In addition, there are freedoms in choosing the form of the equation for u, in particular, to any order, it can be put (at the expense of an appropriate choice of the second equation form) into the form having the same differential structure as the high order KdV equations arising in the unidirectional case but with arbitrary coefficients in the high order differential polynomials.

INTEGRABLE HIGH-ORDER KDV EQUATIONS AND THEIR SOLUTIONS

The freedoms existing in the choice of the equation for the main right-moving wave may be used to make it integrable. For example, it can be chosen to be a member of the KdV family of integrable equations. Those equations have N-soliton solutions which are the same as the KdV N-soliton solutions except for the higher order corrections to the velocity. In the present work, we consider another, more interesting, possibility of constructing exact solutions of the equation for the right-moving wave. It is of interest in that the constructed solutions include impacts of all orders of the expansion and therefore can be treated as a solution of the original problem in a sense. The solutions are constructed using a new approach to the use of the LIe-Bäcklund groups for equations dependent on a small parameter (in the spirit of the approach developed in [4] for point transformations. A central point of the approach is a new integrable equation that is dependent on the parameter β and possesses the following properties: the leading term of its expansion with respect to βrepresents the KdV equation and the higher order terms are differential polynomials of the same weight as those in the corresponding orders of the equation for the right-moving wave. Solutions of the new integrable equations stem, like the solutions of the KdV family of integrable equations, from solutions of the leading order KdV equation but they possess the features that are present neither in the solutions of the leading order KdV equation nor in the solutions of the high-order KdV equations belonging to the KdV family of integrable equations. In particular, a transition from solitons (waves of elevation) to anti-solitons (waves of depression) may occur, when the value of β exceeds some threshold value (this value depends on the soliton parameters and may be rather small). Correspondingly there are solutions that describe an interaction of waves of elevation with waves of depression.

APPLICATIONS IN THE CONTEXT OF ASYMPTOTIC INTEGRABILITY

The new integrable equations constructed via the approach described are also of interest in the context of asymptotic integrability of physical systems.

First, this integrable equation may be used as a reference equation to define conditions for the asymptotic integrability, which provides an approach to the asymptotic integrability problem different from that based on the normal form theory. In the normal form theory, a near identity transformation is sought that converts the original equation at a specific order in β to an equation given by the symmetries of the leading order equation. In the alternative approach, it is required that a near identity transformation applied to the original equation at some order in β convert it to the equation given by the corresponding terms of an asymptotic expansion of the new integrable equation.

Another result obtained with the use of the new integrable equation is that a near identity transformation, applied to define a solution of a high order KdV equation on the basis of the solution of the equation given by the symmetries of the leading order equation, may yield incorrect solutions even for small values of β.

References

[1] Whitham G.B.: Linear and Nonlinear waves. Wiley, NY 1974.

[2] Matioli F. Decomposition of the Boussinesq Equations for Shallow Water Waves into a Set of Coupled Korteweg-de Vries Equations. *Phys. Fluids A* **3**:2355-2359, 1991.

[3] Marshant T.R.: High-Order Interaction of Solitary Waves on Shallow Water. *Studies Appl. Math.* **109**:1-17, 2002.

[4] Burde G.I.: Potential Symmetries of the Nonlinear Wave Equation $u_t = (uu_x)_x$ and related exact and approximate solutions. *J. Phys. A: Math. Gen.* **34**:5355-5371, 2001.