Incremental minimization principles in fracture and damage Mechanics

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The formulation of crack evolution or damage evolution in a body from a principle of minimization of the total energy of the body is an alternative to usual appoaches. It offers as main advantage to be free of any *a priori* assumption on the evolution of the defects in space and in time. On the other hand it is more difficult to introduce into the model the irreversibility condition requiring that the defects can only grow in space with time. A discretization of the loading process and therefore of the evolution is first needed. That leads to the notion of incremental evolution problems. This discrete formulation is sometimes sufficient to obtain accurate information on the damaging process. It is the case when numerical computations are considered. However this bypass is not satisfactory from a theoretical point of view and one has to pass to the limit when the discretization step goes to zero to obtain the so-called "time-continuous evolution problem". From a mathematical point of view this passage to the limit can lead to very technical difficulties. The goal of the conference is to illustrate these different points from several examples chosen in Damage Mechanics, in Griffith Fracture Mechanics and in Barenblatt Fracture Mechanics.

Brittle damage We consider a brittle material the rigidity of which can fail from A_0 to A_1 when a criterion based on the elastic energy release is satisfied: $(A_0 - A_1)\varepsilon \cdot \varepsilon = 2\kappa$, ε denoting the strain tensor and κ is the "toughness" parameter of the material. Considering a body Ω made of this material and submitted to a loading process f_t, the problem consists in finding the characteristic function χ_t of the damaged zone and the displacement field u_t at each instant t. If no irreversibility condition is introduced for the damage evolution (χ_t can return to the value 0 even if it took the value 1 in the past), the evolution problem consists simply in the following minimization:

$$(\chi_t, u_t) = Argmin_{\chi,v} \int_\Omega \Big(\frac{1}{2}((1-\chi)A_0 + \chi A_1)\varepsilon(v)\cdot\varepsilon(v) + \kappa\chi\Big)dx - f_t(v).$$

Because of the loss of lower semicontinuity this problem must be relaxed; there does not exist in general a classical minimizer because the optimum consists in mixing locally the sound and the damaged material to obtain an intermediary effective material the effective elasticity of which is less than the average value of the elasticity of the mixture. The relaxed problem reads then

$$(\alpha_t, u_t) = Argmin_{\alpha,v} \int_\Omega \Big(\frac{1}{2}(A_\alpha)\varepsilon(v)\cdot\varepsilon(v) + \kappa\alpha\Big)dx - f_t(v),$$

where α takes now its value in the whole interval $[0,1]$ and represents the local volume fraction of the damaged material whereas A_α is the optimal homogenized elasticity tensor corresponding to this fraction.

If now we intoduce the irreversibility condition, χ_t must increase with t and the evolution problem is first formulated as the following incremental one: $\chi_0 = 0$ and for $i \geq 1$

$$(\chi_i, u_i) = Argmin_{\chi \geq \chi_{i-1}, v} \int_\Omega \Big(\frac{1}{2}((1-\chi)A_0 + \chi A_1)\varepsilon(v) \cdot \varepsilon(v) + \kappa\chi \Big) dx - f_i(v).$$

Always for the same reasons, this problem must be relaxed. For the first step, the relaxation remains unchanged, but for the next steps one encounters a real difficulty. It is in general not sufficient to consider increasing volume fraction α_t with t, the growth of the minimizing sequences must also be considered.

Griffith Fracture Mechanics The elastic body can now contain growing cracks view as surfaces of discontinuity $S(u)$ of the displacement field u. If we adopt Griffith assumption for the surface energy, denoting by G_c the toughness of the material, the incremental problem reads as : $u_0 = 0$ and for $i \geq 1$

$$u_i = Argmin_{v \in \mathcal{C}_i : S(v) \supset S(u_{i-1})} \int_\Omega \frac{1}{2} A\varepsilon(v) \cdot \varepsilon(v) dx + G_c \mathcal{H}^2(S(v))$$

where $\mathcal{C}_i$ denotes the set of admissible displacements at step i. To obtain the time continuous evolution problem we must pass to the limit when the discretization step goes to zero. This task can be achieved for a scalar problem (like in antiplane elasticity) but, at the present time, not in the general context of threedimensional elasticity for technical reasons. In the limited context where the proof works, the limit problem consists always in a minimization of the energy with respect to "admissible" larger surfaces of discontinuities plus a condition of continuity of the real energy with respect to time.

Barenblatt Fracture Mechanics If the Griffith surface energy is replaced by a Barenblatt surface energy by introducing a continuous increasing function of the jump of the displacements, $\phi([\![u]\!])$, increasing from 0 to G_c and if we do not consider irreversibility of the cracking, the evolution problem reads simply as

$$u_t = Argmin_{v \in \mathcal{C}_t} \int_\Omega \frac{1}{2} A\varepsilon(v) \cdot \varepsilon(v) dx + \int_{S(v)} \phi([\![v]\!]) d\mathcal{H}^2(x).$$

To take into account irreversibility, we introduce an internal variable δ on the surface of discontinuity $S(u)$, called the cumulated opening of the crack, which can only increase with time. The incremental problem traducing the evolution of the cracks is then : $u_0 = 0$ anf for $i \geq 1$

$$u_i = Argmin_{v \in \mathcal{C}_i} \int_\Omega \frac{1}{2} A\varepsilon(v) \cdot \varepsilon(v) dx + \int_{S(v)} \phi(\delta_{i-1} + [\![v - u_{i-1}]\!]^+) d\mathcal{H}^2(x),$$

and $\delta_i = \delta_{i-1} + [\![v - u_{i-1}]\!]^+$, x^+ denoting the positive part of x. It appears that this incremental problem is able to render account for fatigue phenomenon, that is for growth of cracks under cyclic loading.