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APPLICATION OF THE MATERIAL FORCE METHOD TO STRUCTURAL OPTIMIZATION

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Summary The present contribution aims at deriving a variationally consistent strategy to generate truss structures which are optimal in the sense of energy minimization. Accordingly, not only the spatial node point positions of the individual truss members, but also their material node point positions, i.e. the truss geometry itself, are introduced as primary unknowns. The governing equations follow straightforwardly from the Dirichlet principle for conservative mechanical systems. Thereby, the central idea is the reformulation of the total variation of the potential energy at fixed referential coordinates in terms of its variation at fixed material and at fixed spatial coordinates. The corresponding Euler–Lagrange equations define the spatial and the material motion version of the balance of linear momentum, i.e. the balance of spatial and material forces, in a consistent dual format. The suggested algorithm is then essentially characterized through the discretization and simultaneous solution of both, the spatial and the material motion problem. In this sense, the proposed strategy can be interpreted as a variational ALE formulation which renders not only the deformed truss structure but also an improvement of the node point positions themselves. The suggested algorithm will be discussed by means of illustrative examples.

INTRODUCTION

The material force method has gained an increasing interest over the last decade. The notion of material forces, or rather forces on singularities, dates back to the early work of Eshelby [8], in the honor of whom the mechanics on the material manifold have been termed Eshelbian mechanics. While the classical spatial motion problem typically deals with spatial or Newtonian forces which are essentially conjugate to spatial variations at fixed material positions, material or Eshelbian forces are generated by material variations at fixed positions in the spatial configuration. Classically, material forces were introduced in the context of material inhomogeneities,see e.g the textbooks by Maugin [12] and Gurtin [9] or in the recent work of our own group [1, 10, 11, 14, 15]. However, the application of material forces is not only restricted to physical inhomogeneities. In the finite element context, material forces can be utilized, to locate discrete inhomogeneities introduced by the finite element discretization. The first discussion on discrete material forces dates back to Braun [5], where, for the first time, finite element nodes were interpreted as discrete defects inducing discrete material forces. It was found, that meshing can be improved by moving the nodes in the direction opposite to the corresponding material forces and thus releasing an additional amount of energy, see Mueller & Maugin [13], Askes et al. [1], Kuhl et al. [10], Thoutireddy & Ortiz [16] and Thoutireddy [17]. An alternative application of the material force method was presented recently by Braun [6] who made use of discrete material node point forces to generate optimal truss structures. In the present work, we shall extend these ideas to the geometrically nonlinear theory and elaborate different solution strategies.

GOVERNING EQUATIONS

The underlying relaxation of the material node point position is strongly related to Arbitrary Lagrangian-Eulerian formulations which have traditionally been applied in the context of moving interfaces, e.g. in fluid-structure-interaction. An ALE formulation is basically characterized through the introduction of an independent fixed reference domain $\mathcal{B}_\square$ next to the classical material and spatial domain $\mathcal{B}_0$ and $\mathcal{B}_t$, see e.g. [1–4, 7, 10]. The following considerations are essentially based on two independent mappings, i.e. the referential maps from the referential to the spatial configuration $\boldsymbol{x} = \bar{\varphi}\,(\,\xi, t\,)\; : \mathcal{B}_\square \to \mathcal{B}_t$ and from the referential to the material configuration $\boldsymbol{X} = \tilde{\boldsymbol{\Phi}}\,(\,\xi, t\,)\; : \mathcal{B}_\square \to \mathcal{B}_0$, with the corresponding deformation gradients $\bar{\boldsymbol{F}} = \nabla_\xi \bar{\varphi}$ and $\tilde{\boldsymbol{f}} = \nabla_\xi \tilde{\boldsymbol{\Phi}}$. In what follows, we shall restrict ourselves to the hyperelastostatic case for conservative systems for which the Dirichlet principle defines the appropriate variational setting. In this context, the conservative mechanical system is essentially characterized through the infimum of the total energy $\mathcal{I}$ or, alternatively, through its vanishing total variation.

$$\mathcal{I}\,(\,\bar{\varphi}, \tilde{\boldsymbol{\Phi}}\,) = \int_{\mathcal{B}_\square} W_\square + V_\square \;\mathrm{d}V_\square \to \inf \qquad \delta\mathcal{I} = \delta_{\boldsymbol{x}}\mathcal{I} + \delta_{\boldsymbol{X}}\mathcal{I} \doteq 0 \tag{1}$$

Thereby $\mathcal{I}$ can be understood as the integral over the internal potential energy density $W_\square$ plus the external potential energy density for conservative systems $V_\square$ over the reference domain $\mathcal{B}_\square$. Note, that in equation (1), the total variation with respect to fixed reference coordinates has been reformulated as the sum of the variation with respect to the spatial coordinates at fixed material positions plus the variation with respect to the material coordinates at fixed spatial position.

DISCRETIZATION

The reference domain can then be discretized in n_{el} linear truss elements. Each truss is interpolated by the linear shape functions in terms of the discrete node point positions ξ_i of the $i = 1, 2$ truss nodes. Following the isoparametric concept, the unknowns $\bar{\varphi}$ and $\tilde{\boldsymbol{\Phi}}$ and their variations $\delta\bar{\varphi}$ and $\delta\tilde{\boldsymbol{\Phi}}$ are interpolated on the element level with the same shape functions N as the element geometry. Recall, that the spatial and the material stretch with respect to the reference truss element of unit length are simply given through the corresponding deformation gradients $\bar{F} = l$ and $\tilde{f} = L$, i.e. the length of the spatial and the material truss element. The discrete spatial and material variation of the total energy $\mathcal{I}$ take the following format

$$\delta_{\boldsymbol{x}}\mathcal{I} = \sum_{I=1}^{n_{\text{nd}}} \delta\bar{\varphi}_I \cdot \boldsymbol{R}^I_{\bar{\varphi}} \qquad \delta_{\boldsymbol{X}}\mathcal{I} = \sum_{J=1}^{n_{\text{nd}}} \delta\tilde{\boldsymbol{\Phi}}_J \cdot \boldsymbol{R}^J_{\tilde{\boldsymbol{\Phi}}} \tag{2}$$

with the discrete residuals of the spatial and the material motion problem $\boldsymbol{R}^I_{\bar{\varphi}}$ and $\boldsymbol{R}^J_{\tilde{\boldsymbol{\Phi}}}$ defined in the following form.

$$\boldsymbol{R}^I_{\bar{\varphi}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_i \, \Pi \, A \, \boldsymbol{n}_e \doteq \mathbf{0} \qquad \boldsymbol{R}^J_{\tilde{\boldsymbol{\Phi}}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_j \, \pi \, A \, \boldsymbol{N}_e \doteq \mathbf{0} \tag{3}$$

LINEARIZATION

The above derived discrete residual statements represent a highly nonlinear coupled system of equations which can be solved within the framework of a monolithic incremental iterative Newton–Raphson solution strategy. To this end, we perform a consistent linearization of the governing equations which introduces the following contributions to the overall iteration matrix.

$$\begin{aligned}
\boldsymbol{K}^{IK}_{\bar{\varphi}\bar{\varphi}} &= \mathrm{D}_{\bar{\varphi}_K} \boldsymbol{R}^I_{\bar{\varphi}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_I \, \mathrm{D}_{\bar{F}} \Pi \, A \, \boldsymbol{n}_e \otimes \boldsymbol{n}_e \, \nabla_\xi N_K + \nabla_\xi N_I \, \Pi \, A \, \frac{1}{l} \, [\, \boldsymbol{I} - \boldsymbol{n}_e \otimes \boldsymbol{n}_e \,] \, \nabla_\xi N_K \\
\boldsymbol{K}^{IL}_{\bar{\varphi}\tilde{\boldsymbol{\Phi}}} &= \mathrm{D}_{\tilde{\boldsymbol{\Phi}}_L} \boldsymbol{R}^I_{\bar{\varphi}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_I \, \mathrm{d}_{\tilde{f}} \Pi \, A \, \boldsymbol{n}_e \otimes \boldsymbol{N}_e \, \nabla_\xi N_L \\
\boldsymbol{K}^{JK}_{\tilde{\boldsymbol{\Phi}}\bar{\varphi}} &= \mathrm{D}_{\bar{\varphi}_K} \boldsymbol{R}^J_{\tilde{\boldsymbol{\Phi}}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_I \, \mathrm{D}_{\bar{F}} \pi \, A \, \boldsymbol{N}_e \otimes \boldsymbol{n}_e \, \nabla_\xi N_K \\
\boldsymbol{K}^{JL}_{\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}} &= \mathrm{D}_{\tilde{\boldsymbol{\Phi}}_L} \boldsymbol{R}^J_{\tilde{\boldsymbol{\Phi}}} = \mathop{\mathbf{A}}_{e=1}^{n_{\text{el}}} \nabla_\xi N_J \, \mathrm{d}_{\tilde{f}} \pi \, A \, \boldsymbol{N}_e \otimes \boldsymbol{N}_e \, \nabla_\xi N_L + \nabla_\xi N_J \, \pi \, A \, \frac{1}{L} \, [\, \boldsymbol{I} - \boldsymbol{N}_e \otimes \boldsymbol{N}_e \,] \, \nabla_\xi N_L
\end{aligned} \tag{4}$$

The solution of the linearized system of equations finally not only renders the iterative update for the deformed configuration $\bar{\varphi}_K = \bar{\varphi}_K + \mathrm{d}\bar{\varphi}_K$ but also the iterative update for the optimal truss structure $\tilde{\boldsymbol{\Phi}}_L = \tilde{\boldsymbol{\Phi}}_L + \mathrm{d}\tilde{\boldsymbol{\Phi}}_L$. Recall, that alternatively, a staggered solution technique can be applied, which follows straightforwardly from the above algorithm by neglecting the coupling terms $\boldsymbol{K}^{IL}_{\bar{\varphi}\tilde{\boldsymbol{\Phi}}}$ and $\boldsymbol{K}^{JK}_{\tilde{\boldsymbol{\Phi}}\bar{\varphi}}$.

CONCLUSIONS

The key idea of the present contribution was the application of the material force method for the optimization of truss structures. Motivated by a variational Arbitrary Lagrangian Eulerian formulation in combination with a particular parametrization, we derived the spatial and the material motion version of the balance of linear momentum. The solution of the former renders the spatial configuration while the latter defines the related material configuration, i.e. node point positions of the structural elements which are optimal in the sense of energy minimization.

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