NONLINEAR OSCILLATORS WITH TIME DELAYS

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Summary: A nonlinear damped oscillator, with a small part of the restoring force retarded in time is considered. It appears that the dynamics of such a system becomes quite complex. Depending on the retardation time τ a number of Hopf bifurcations as well as a number of various chaotic regimes exist. The equation considered here was invented as the simplest model of keyhole instabilities observed during the laser welding.

A delay differential equation is proposed to model the dynamics of a capillary crated in the molten metal by the laser beam in the process of laser welding. The capillary, called also keyhole, is filled with ionized metallic vapours. The pressure of these vapours acts against the forces of surface tension and prevents the capillary from the collapse [1,2,3,5]. In experiments, the deterministic chaos has been detected in the laser welding process [6,7].
By introducing the oscillation frequency ω of the linearized equation in the vicinity of the stable equilibrium point r_0, taking into account only radial oscillations, one can write after some simplifications the equation describing the radial motion of the keyhole capillary walls in the following form:

(1) $$\ddot{r} + \gamma\,\dot{r} = -\omega^2 \frac{(r - r_1)(r - r_0)}{r^3}, \qquad 0 < r_1 < r_0,$$

where γ represents dampening, proportional to the viscosity of the liquid. and r_1, r_0 are depending on the laser power. Obviously Eq. (1) does not produce self-excited oscillations and moreover self-sustained chaotic oscillations, which are observed in the experiment. However the ionised vapour expanding from the channel forms a plasma plume above the welded surface. This plasma plume absorbs part of laser beam energy and radiates it away. Additionally, the refraction of the laser radiation in the plasma plume results in defocusing of the laser beam causing further decrease of the power reaching the capillary. One of the reasons for the appearance of chaotic fluctuations seems to be the time delay between the variations of the pressure in the keyhole capillary and changes of the size and density of the plasma plume, resulting in the variations of the energy transfer from laser beam to the material. The size of the plasma plume is a function of the channel radius history, which may be represented here by a certain delay time τ. We can assume that the laser power reaching the keyhole is given in the first approximation by the linear expression $P = P_0 + \alpha\, r(t-\tau) + \ldots$ where α is some small parameter. Consequently up to higher order terms in the expansion into power series one can postulate $r_0 = a + \alpha\, r(t-\tau) + \ldots$. Taking the period of the linearized oscillations as a time unit ($T = 2\pi\,\omega^{-1}$ for $\alpha = 0$) and introducing the normalised channel radius $x(t) = r(t)/a$, and finally, neglecting radius r_1 as a small value in Eq. (1) one arrives at the following delay differential equation

(2) $$\ddot{x} + \gamma\,\dot{x} + \frac{1}{x^2}\{(x-1) + \alpha(x_r - 1)\} = 0, \qquad \text{where} \qquad x_r = x(t-\tau).$$

Linearizing our equation around equilibrium state $x \equiv 1$ we obtain a simple linear damped oscillator perturbed by a linear force terms retarded in time. The characteristic quasi-polynomial obtained by inserting $y = e^{\lambda t}$ for the solution of this equation is

(3) $$\lambda^2 + \gamma\lambda + 1 + \alpha e^{-\lambda\tau} = 0.$$

For further analysis let us decompose $\lambda = p + iq$. We are interested in possible Hopf bifurcations, i.e. in the purely imaginary solutions of Eq. (3). When $p = 0$, the equation (3) can be written in the following form

(4) $$\cos^2 q\tau + \frac{\gamma^2}{\alpha}\cos q\tau + \left(\frac{\gamma^2}{\alpha^2} - 1\right) = 0, \qquad q = \frac{\alpha}{\gamma}\sin q\tau,$$

In particular, for small γ/α and small $\alpha < 1$ the approximate solutions of Eqs (4) can be given and finally we have the following values of τ for which Hopf bifurcation takes place:

(5) $$\tau_k^+ = \frac{\gamma}{\alpha} + \frac{2\pi k}{\sqrt{1+\alpha}} + \ldots \qquad \tau_k^- = \pi - \frac{\gamma}{\alpha} + \frac{2\pi k}{\sqrt{1-\alpha}} + \ldots, \qquad k = 0, 1, 2, \ldots$$

Let us notice that the points $\tau_k^{\pm}$ are the values of τ at which the curve $p=p(\tau)$ satisfying equations (3) crosses the axis of τ. Since for τ close to zero the stationary solution is stable, therefore, up to the first point of bifurcation $\tau_0^+ = \gamma/\alpha$, the value of $p(\tau)$ is negative and it changes sign for $\tau > \gamma/\alpha$. Then again after passing the next point τ_0^-, $p(\tau)$ becomes negative, which implies the stability of the stationary solution $x \equiv 1$. In this way we deduce that the stationary solution is stable for $\tau \in \left(\tau_{k-1}^-, \tau_k^+\right)$ and unstable for $\tau \in \left(\tau_k^+, \tau_k^-\right)$, where $k = 0,1,2,3,...$ and $\tau_{-1}^- = 0$. In the first approximation the corresponding frequency is not dependent on k and is equal to $q_{\pm} = \sqrt{1 \pm \alpha}$, respectively.

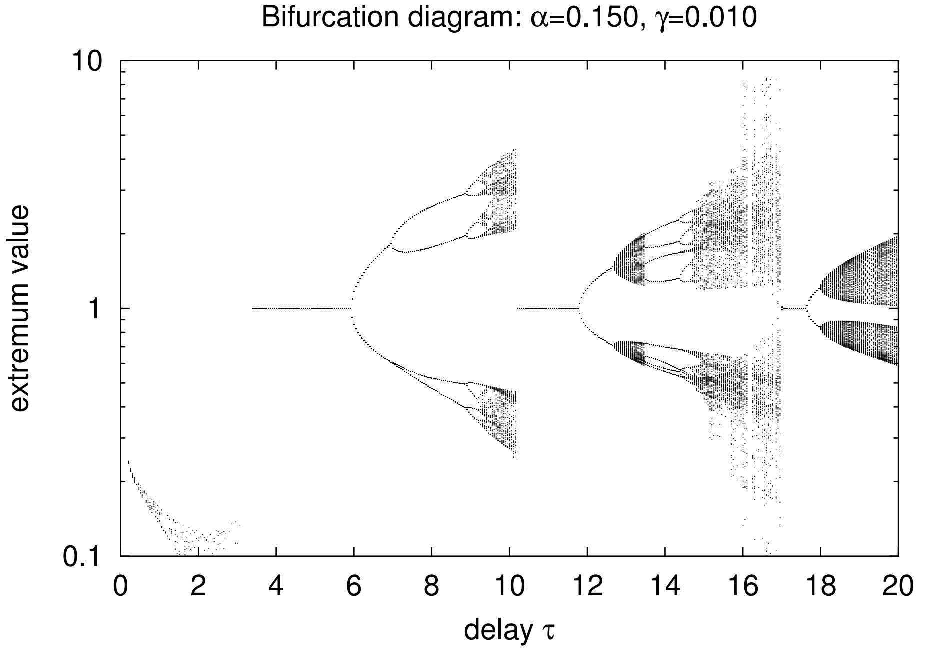

Fig. 1. Bifurcation diagram as a function of time delay τ, for γ=0.01 and α=0.15, resulting from the numerical integration of Eq. (2). One can notice the transitions to chaos by cascades of period doubling as well as through the torus breakdown (for larger values of τ). Please note a good agreement of bifurcation points with theoretical predictions.

Conclusions. Although the numerical simulation presented here were performed for quite specific equation (Eq. (2)) , our experience shows that qualitatively other nonlinearities lead to similar behavior. Clearly, our analytical bifurcation analysis is in fact not dependent on the nonlinearity as long as there exists an equilibrium state for every $\tau \geq 0$ which is asymptotically stable for $\tau = 0$. Some part of results presented here can also be given for systems of delay differential equations. Thus possible area of applications can be much wider then the presented here example.
Finally, as it follows from our analytical considerations and also from the numerical simulations, the Hamiltonian systems or Hamiltonian systems with weak dissipative terms can exhibit drastically different dynamics when slightly polluted with the time delay terms.

References

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