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SECONDARY BIFURCATIONS AND LOCALISATION OF BUCKLE PATTERNS

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Summary The phenomenon of localisation is reasonably well understood in quantum physics and fluid mechanics [1], but less so in the area of structural mechanics. In this contribution we revisit the effect of secondary bifurcations on the post-buckling response of a 3D system of elastically restrained beams. Our objective is to construct a uniform asymptotic expansions for the localised buckling patterns experienced by this model by using a mixture of asymptotics (WKB techniques [2]) and numerics.

INTRODUCTION AND THEORETICAL STUDY

Many buckling problems for structures with large aspect ratio can be cast as

$$\varepsilon \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{A}(x,\ \lambda;\ \varepsilon)\mathbf{u}, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(x) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ represents the bifurcation parameter, and $\mathbf{A}(x,\ \lambda;\ \varepsilon)$ is an $n \times n$ matrix, i.e., $\mathbf{A}(x,\ \lambda;\ \varepsilon) \in M_{n\times n}(\mathbb{R})$; here $0 < \varepsilon \ll 1$ is a small parameter. The dependence of $\mathbf{A}$ on x is usually associated with variable mechanical/geometrical properties, or non-homogeneous pre-bifurcation states; certain secondary bifurcations usually fall into this latter category, as we shall shortly see. The presence of inhomogeneities in equation (1) enriches the spectral properties of the problem and calls for a special attention to certain transition points (also known as *turning points*). Such points mark the transition between oscillatory and exponentially decaying regions in the solutions of (1), and in particular they are responsible for the presence of localisation in these functions.

The model

The model adopted in this work is that discussed by Luongo in [3] (see also [4] for a related problem) and which we briefly review next. Roughly speaking, we have in mind a simply supported planar truss consisting of two identical horizontal beams connected by bars. The two beams are restrained against out-of-plane displacements by linear elastic springs; the linking bars are assumed rigid in the xy–plane and infinitely flexible out-of-plane (see Figure 1). To a certain extent, the model mimics the behaviour of thin-walled beams in compression in the sense that the system can buckle either in an overall mode (a “long-wave" in the xy–plane) or in a local mode (a “short-wave" out-of-plane). It has been known for a long time that the extreme imperfection-sensitivity of shell structures is mostly due to an interaction between linearly independent simultaneous, or nearly simultaneous, such global and local buckling modes.

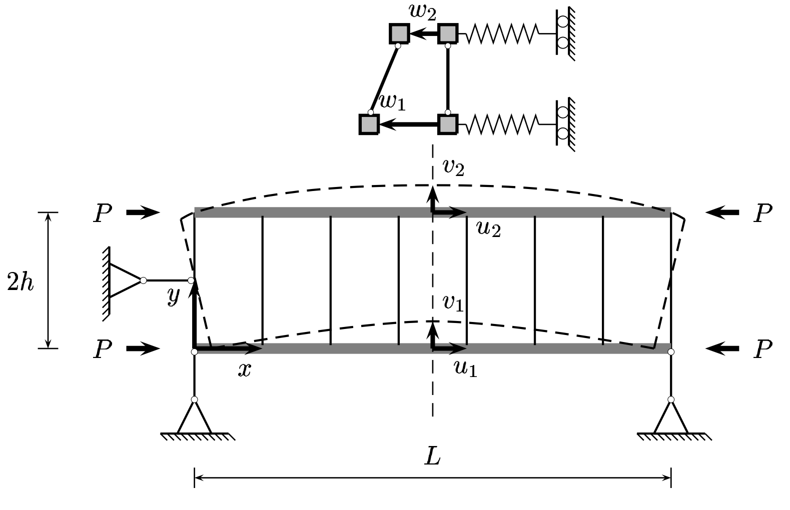

Figure 1. A system of two elastically restrained beams: lateral view (bottom) and axial perspective (top).

Governing equation and asymptotic analysis

By taking into account the long-wave buckling mode and assuming that the overall and local buckling modes correspond to closely spaced load values, one finds that the local instability is described by the (rescaled) boundary-value problem

$$\varepsilon^4 \frac{d^4 w}{dx^4} + 2\varepsilon^2 \left[(1-\gamma)\frac{d^2 w}{dx^2} + \lambda \frac{d}{dx}\left(\sin(x)\frac{dw}{dx}\right)\right] + w = 0, \qquad x \in (0,\ \pi), \tag{2a}$$

$$w = \frac{d^2 w}{dx^2} = 0, \qquad \text{for} \quad x = 0,\ \pi, \tag{2b}$$

where w is the out-of-plane displacement associated with the bottom beam. Here ε is a parameter depending on the geometrical and mechanical properties of the truss, λ corresponds to a rescaled loading parameter, while γ accounts for the closeness of the critical loads corresponding to the two modes of instability. It can be shown that the turning points for (2) are the two solutions x_1 and x_2 in $(0,\,\pi)$ of the equation $\sin(x) = \gamma/\lambda$. A solution of (2) is sought in the form

$$w(x,\,\varepsilon) = W(x)\exp\left\{i\left(\varepsilon^{-1/2}\eta_0\zeta + \frac{1}{2}a\zeta^2\right)\right\}, \tag{3}$$

where $W(x) = W_0(x) + \varepsilon^{1/2}\,W_1(x) + \varepsilon\,W_2(x) + \ldots$, $\lambda = \lambda_0 + \varepsilon\,\lambda_1 + \varepsilon^2\,\lambda_2 + \ldots$, and $\zeta = \varepsilon^{-1/2}(x - x_0)$. Above $x_0 \in (0,\,\pi)$ is supposed to be the centre of localisation, the point where most of the energy is concentrated. Parameter η_0 is meant to characterise the oscillation frequency of the spatial eigenmode, while $\mathfrak{I}(a)$, the imaginary part of $a \in \mathbb{C}$, gives the decay rate of the solution amplitude. Thus, localisation requires η_0 to be real and $\mathfrak{I}(a) > 0$; also, we assume tacitly that x_1 and x_2 are close to each other, precisely $|x_1 - x_2| \sim \varepsilon^{1/2}$. Standard calculations carried out in [5], lead to the determination of all unknowns in (3).

NUMERICAL EXPERIMENTS

It is found that the eigenvalues of (2) are given by $\lambda^{(m)} = \gamma + \varepsilon\sqrt{\gamma}\,(1+2m) + \mathcal{O}(\varepsilon^2)$, where the lowest eigenvalue corresponds to $m = 0$. These eigenvalues are asymptotically double and to each of them there correspond two eigenmodes

$$u^{(0,j)}(x,\,\varepsilon) = \left[\cos\left(\frac{2x-\pi}{2\varepsilon} + \Psi_{0j}\right) + \mathcal{O}(\varepsilon^{1/2})\right]\exp\left\{-\frac{\sqrt{\gamma}\,(2x-\pi)^2}{16\varepsilon}\right\}, \tag{4}$$

where $j = 1,\,2$; the phase-angles Ψ_{0j} are found by symmetry considerations as $\Psi_{00} = \pi$ (symmetric mode) and $\Psi_{01} = \pi/2$ (antisymmetric mode). Two typical comparisons between this formula and our numerical runs of (2) are included below. The core of the buckling patterns is described quite satisfactorily by (4), although the tails of the solution in (b)

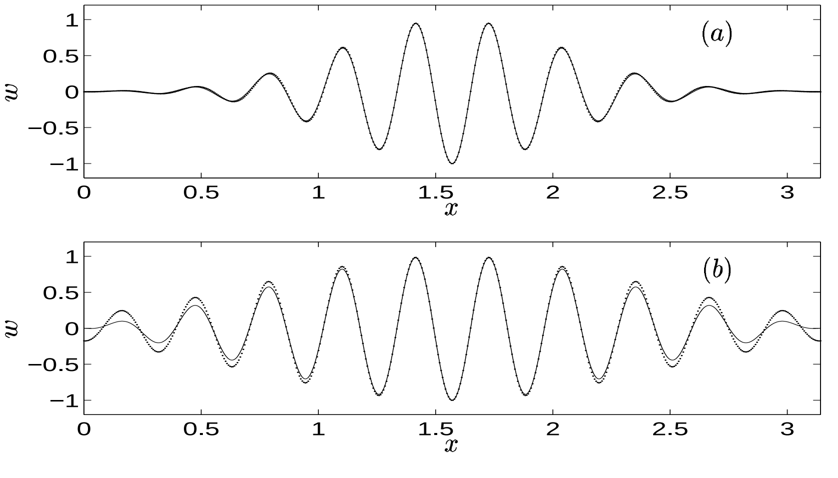

Figure 2. Comparisons between numerical solutions (solid line) and the asymptotic predictions (dotted line) for the first symmetric eigenmodes given by (4); (a) corresponds to $\gamma = 0.2$, while (b) is obtained for $\gamma = 0.02$. In both pictures $\varepsilon = 0.05$.

are poorly approximated. It turns out that this is due to the fact that the scaling assumption for the distance between the two turning points of (2) is violated in Figure 2(b); details on this subtle point are provided in [5].

CONCLUSIONS

In contrast to other works on similar topics [3, 4], the approach taken here succeeds in producing a compact uniform asymptotic expansion for the buckling patterns, valid over the entire span of the truss. We believe that more complex solid mechanics problems involving secondary bifurcations are susceptible to a similar analysis, and this is currently under investigation.

References

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