NON-LINEAR OSCILLATOR UNDER RANDOM RENEWAL-DRIVEN TRAINS OF IMPULSES

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Summary A non-linear, non-hysteretic oscillator under a random train of impulses driven by a class of renewal processes is considered. The original state vector of the oscillator is augmented by a number of auxiliary state variables. The augmented state vector is governed by two independent Poisson processes, hence the original non-Markov problem is converted into a Markov one. Response mean value and variance are obtained from the equations for moments and are verified against Monte Carlo simulations.

INTRODUCTION

Random pulse trains are valid models of actual excitations processes such as e.g. irregular trains of shocks and impacts. If the dynamic system is acted upon by a non-Poisson distributed train of impulses and/or the pulses have general shapes, the state vector is not a Markov process. Various techniques have been developed to convert original non-Markov pulse problems into Markov ones, for example for trains of impulses driven by Erlang renewal processes [1], or for sine half-wave Poisson-driven pulses [2]. A random train of impulses with interarrival time being a sum of two independent, negative exponential distributed random variables with different parameters (generalized Erlang renewal process) was dealt with in [3]. The excitation process was exactly expressed in terms of an auxiliary variable, governed by a stochastic equation driven by two independent Poisson processes with different parameters.

In the present paper a non-linear, non-hysteretic oscillator under a random train of impulses driven by a renewal process is considered. The class of renewal impulse processes considered is obtained by multiplying the impulse magnitudes of an Erlang renewal impulse process by the values of an intermittent, zero-one auxiliary stochastic variable. This variable is governed by a stochastic differential equation driven by two independent Erlang renewal processes, each of which is exactly expressed, with the aid of a set of auxiliary variables, in terms of a Poisson process. Thus the augmented state vector, consisting of the original state vector and of auxiliary variables, is driven by two independent Poisson process, and becomes a Markov process. The Ito's differential rule is used to derive the differential equations governing the response statistical moments. The usual and special cumulant-neglect closure techniques are used to truncate the hierarchy of moments equations. The mean value and variance of the response are obtained by numerical integration of a suitably truncated set of moment equations and verified against Monte Carlo simulations.

STATEMENT OF THE PROBLEM

Consider a non-linear, non-hysteretic oscillator governed by the equation

$$\ddot{X}(t) + f\big(X(t), \dot{X}(t)\big) = \sum_{i,R=1}^{R(t)} P_{i,R}\,\delta(t - t_{i,R}), \tag{1}$$

where $f\big(X(t), \dot{X}(t)\big)$ is the function of the instantaneous values of $X(t)$ and $\dot{X}(t)$ and the stochastic excitation is the random train of impulses whose arrival times $t_{i,R}$ are driven by the renewal process $R(t)$. Sample functions of counting processes are assumed herein to be left-continuous with right limit. The impulses magnitudes $P_{i,R}$ are given by independent random variables with common probability density function.

The renewal driven train of impulses in (1) may be represented as follows

$$\sum_{i,R=1}^{R(t)} P_{i,R}\,\delta(t - t_{i,R}) = \sum_{i=1}^{R_\nu(t)} Z(t_i) P_i\,\delta(t - t_i), \tag{2}$$

where the arrival times t_i are driven by an Erlang renewal process $R_\nu(t)$ with parameters ν and k and $Z(t_i)$ is a value at t_{i-} of an intermittent, zero-one stochastic variable $Z(t)$ governed by the stochastic equation

$$dZ(t) = (1 - Z)\,dR_\mu(t) - Z\,dR_\nu(t), \tag{3}$$

where $R_\mu(t)$ is an Erlang renewal process with parameters μ and l. The Erlang process $R_\nu(t)$ and $R_\mu(t)$ are assumed to be independent. The variable $Z(t)$ equals zero except in the time interval between the first $R_\mu(t)$ driven event occurring after an $R_\nu(t)$ driven event and the first subsequent $R_\nu(t)$ driven event. In other words, $Z(t_i)$ is zero at all instants t_i driven by $R_\nu(t)$, except the first ones occurring after $R_\mu(t)$ driven events. The equation (3) is a generalisation of an equation driven by two independent Poisson processes given in [4].

The increment of the underlying renewal process $R(t)$ equals, with probability 1, $dR(t) = Z(t)dR_\nu(t)$, which follows from the fact that both counting processes are regular (the increments only take values 0 or 1).

An Erlang renewal process $R_\alpha(t)$ ($\alpha = \mu, \nu$) may be expressed in terms of the Poisson process $N_\alpha(t)$ ($\alpha = \mu, \nu$) at the expense of introducing auxiliary variables, for any integer parameter k or l [5]. For any α the following replacement is valid

$$dR_\alpha(t) = \rho_\alpha(t) dN_\alpha(t), \quad \alpha = \mu, \nu \tag{4}$$

where the $\rho_\alpha(t)$ is a variable which only takes values 0 or 1 and is expressed in terms of a number of further, discrete-valued, auxiliary variables which are governed by stochastic differential equations driven by the Poisson process.

SOLUTION TECHNIQUE

The augmented state vector $\mathbf{X}(t)$ consisting of $X(t)$, $\dot{X}(t)$, $Z(t)$ and other auxiliary variables is governed by the stochastic equation

$$d\mathbf{X}(t) = \mathbf{c}\Big(\mathbf{X}(t)\Big)dt + \mathbf{b}\Big(P(t), \mathbf{X}(t)\Big)d\mathbf{N}(t), \quad d\mathbf{N}(t) = \begin{bmatrix} dN_\nu(t) \\ dN_\mu(t) \end{bmatrix}, \tag{5}$$

hence $\mathbf{X}(t)$ is a non-diffusive Markov process.

Differential equations governing the response moments (statistical moments of the state variables $\mathbf{X}(t)$) are obtained from the generalized Itô's differential rule [5], [6]. For the non-linear oscillators with terms $f\left(X(t), \dot{X}(t)\right)$ which are polynomials in $X(t)$ and $\dot{X}(t)$, the right-hand sides of the equations for moments involve the unknown expectations of the non-linear transformations of state variables . The equations for moments form an infinite hierarchy and cannot be directly solved. These unknown expectations, or moments, can only be evaluated approximately, using suitable closure approximations. In the present paper two closure techniques are used. The first one is the ordinary cumulant-neglect closure. The second one is a special closure technique developed for the present problem. The latter closure technique is obtained from the tentative joint probability density of the state vector, assumed in form of the sum of a discrete and a continuous part. The discrete part accounts for the fact that is the system is at rest at the initial time instant, it is still at rest before the first impulse occurs. The continuous part is constructed from two kinds of conditional probability densities, conditioned on the states 0 and 1 of the variable $Z(t)$. Then the ordinary cumulant-neglect closure approximations are applied to conditional moments only, yielding the modified closure approximations for the unconditional moments of the present problem.

The mean value and variance of the response are obtained by numerical integration of a suitably truncated set of moment equations and verified against direct Monte Carlo simulations.

CONCLUSIONS

For a non-linear, non-hysteretic oscillator under a renewal impulse process stochastic excitation a technique is devised to convert the original non-Markov problem into a Markov one. This is done at the expense of introducing auxiliary, discrete-valued state variables. A special cumulant-neglect closure techniques is devised to truncate the hierarchy of moments equations. This closure technique as well as the ordinary one are used in order to evaluate the mean value and variance of the response. The approximate analytical results are verified against direct Monte Carlo simulations.

References

[1] Iwankiewicz R., Nielsen S.R.K.: Dynamic response of non-linear systems to renewal impulses by path integration. *J. Sound Vib Mech* **195**:175–193, 1996.

[2] Ricciardi G.: Random vibration of a beam under moving load. *J. Engrg Mech* **120**, 1994.

[3] Iwankiewicz R.: Dynamic systems under random impulses driven by a generalized Erlang renewal process. In: Proceedings of the 10th IFIP WIG 7.5 Working Conference on Reliability and Optimization of Structural Systems, 25-27 March 2002, Kansai University, Osaka, Japan. Eds. H.Furuta, M.Dogaki and M.Sakano, Balkema 2003,103-110.

[4] Iwankiewicz R.: Dynamic response of non-linear systems to random trains of non-overlapping pulses. *Meccanica*, **37**:167–178, 2002.

[5] Iwankiewicz R., Nielsen, S.R.K.: Vibration Theory, Vol. 4, Advanced methods in stochastic dynamics of non-linear systems, Aalborg University Press, Denmark, 1999.

[6] Snyder, D.L.: Random point processes, John Wiley, New York, 1975.