Slow Rotation of a Double Sphere in a Viscous Fluid

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Estimating the couple acting on a solid suspended in a viscous fluid is a well-known old problem in hydrodynamics. As explained in the literature, the analytical treatments for such boundary value problems become quite cumbersome for nonspherical bodies owing to their complex geometrical structures. In fact, there is no standard method available todate which could yield solutions for a body of arbitrary profile. Despite this fact, several closed form analytic solutions have been discovered during the last 150 years (see for instance [1]). Although the modern computing schemes could handle a variety of problems, it is of important to have convincing results to support and test the relevent numerical codes. Here, we provide an exact closed form solution for a nonspherical body rotating in Stokesian fluid.

We consider two spheres of radii 'a' and 'b' centered at positions A and B respectively. The two spheres overlap and intersect at an angle $\frac{\pi}{n}, n$ an integer. The distance 'c' between the centers is $c = \left[a^2 + b^2 + 2ab\cos\frac{\pi}{n}\right]^{1/2}$. The composite geometry consisting of two overlapping spheres is called a 'double sphere'. Let A_j, B_j be the successive inverse points lying along the line joining the centers. The first point A_1 is the image of B in sphere A and B_1 is the image of A in sphere B etc. The distances $a_j = AA_j$ and $b_j = BB_j$ satisfy the recurrence relations $a_j = \dfrac{a^2}{c - b_{j-1}}$, $b_j = \dfrac{b^2}{c - a_{j-1}}$ with initial values $a_0 = b_0 = 0$. For the angle of intersection $\frac{\pi}{n}$, with n an integer, one can prove that [2] $a_{n-1} + b_{n-1} = c$. In this case the image points A_{n-1} and B_{n-1} coincide. It can be further shown that the distances A_jP and B_jP (P is the vertex) can be expressed in terms of a_j and b_j by

$$A_jP = \left[a_j^2 + \frac{b^2 - a^2 - c^2}{c}a_j + a^2\right]^{1/2}, \qquad B_jP = \left[b_j^2 + \frac{a^2 - b^2 - c^2}{c}b_j + b^2\right]^{1/2} \tag{1}$$

Let r, r' denote the distances of an arbitrary point with A and B as origins respectively. Similarly, let r_j, r'_j denote the distances of the same point with A_j and B_j as origins. The surface of the double sphere is composed of spheres A and B defined by $r = a$ and $r' = b$, respectively.

It is known that for the axial rotation of an axisymmetry body the function $\Psi(R, \phi, z) = v_\phi(R, z)\sin\phi$, where v_ϕ is the tangential component of velocity, satisfies the Laplace equation $\nabla^2\Psi = 0$. The corresponding boundary value problem is well documented elsewhere [3] and therefore we omit the details here. We now exploit the fact that for a sphere rotating about a diameter in a Stokesian fluid the image is just a rotlet/couplet, the Green's function for rotational flow, located at its center. For two spheres A and B the similar rotation produces two rotlets at their centers. The successive reflections of each rotlet on either sphere generates a series of rotlets on the line of centers and for the vertex angle $\frac{\pi}{n}$, by a direct computation it can be demonstrated that, the resulting series is finite. The final solution for the tangential velocity component reads

$$v_\phi(R, z) = -\omega R\left\{\frac{a^3}{r^3} + \frac{b^3}{r'^2} - \sum_{j-1}^{n-1}{}'(-1)^{j+1}\left[\frac{A_jP^3}{r_j^3} + \frac{B_jP^3}{r'^3_j}\right]\right\}. \tag{2}$$

where $r^2 = R^2 + z^2$ etc. and the prime on the summation indicates that the last term must be divided by 2. It is evident that the image system consists of rotlets of strengths -ωa^3, -ωb^3, $(-1)^{j+1}\omega A_jP^3$ and $(-1)^{j+1}\omega B_jP^3$ located at A, B, A_j and B_j respectively. The distances A_jP and B_jP may be calculated by the use of the relation (1). The image rotlets also depend on the magnitude of the angular velocity ω. It is now straightforward to extract an expression for the torque/couple from the singularity solution (2), which

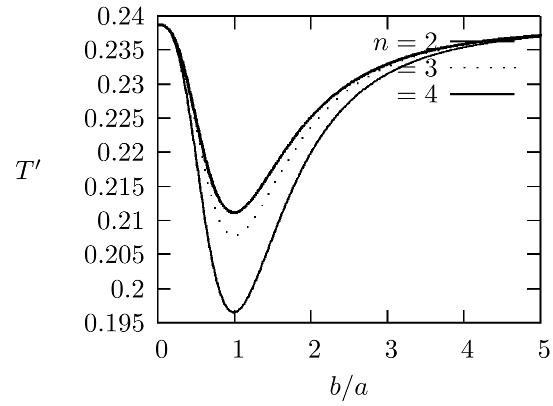

Figure 1: The reduced couple $T' = \frac{T}{8\pi\mu\omega(V_a+V_b)}$ versus b/a for different vertex angles.

is given by

$$T = 8\pi\mu\omega\left[a^3 + b^3 - \sum_{j=1}^{n-1}{}'(-1)^{j+1}(A_jP^3 + B_jP^3)\right]. \tag{3}$$

For $b = 0$, the above expression yields the familiar Kirchhoff's result for an isolated sphere rotating in a viscous fluid. For larger n it can be shown numerically that (3) approaches the value for two touching spheres. The reduced torque $T' = \frac{T}{8\pi\mu\omega(V_a+V_b)}$ (V_a and V_b are the volumes of the spheres A and B) is plotted against the radii ratio b/a in Fig.1 for various vertex angles. It is readily seen that the reduced torque decreases for $b/a < 1$ until it reaches its minimum value. For the values $b/a > 1$, it increases gradually till it becomes a constant. It may be noted that the contact angle has significant influence on the minimum value of the reduced torque. Now, it is not difficult to show that the quantity $\frac{T}{8\pi\mu\omega V_a}$ is always $\geq$ unity. In view of this, it may be conjectured that

$$V_a \leq \frac{T}{8\pi\mu\omega} \leq V_a + V_b$$

The above inequality gives the upper and lower bounds for the torque experienced by the double sphere rotating in a viscous fluid. It should be mentioned that for arbitrary vertex angle, the problem is solved by the use of toroidal coordinates [3] in terms of complicated Mehler conal functions. The corresponding expression for the torque is also given as a quadrature. Our method clearly avoided the use of such special functions and indeed our results could be used to examine those quadratures. Needless to say that the present approach may conveniently be extended to solve other swirling flow problems involving intersecting spheres and their interaction with other particles; however the use of toroidal coordinates method for such flow problems remains to be explored. It is worthmentioning that the results provided here may be useful in validating the numerical algorithms for merging objects [4] in electrostatic as well as in microfluidic environments.

References

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