ON THE IMPACT LAW IN ELASTIC PLATE-LIKE BODIES

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Summary. We study the problem of normal impact of a rigid sphere on a circular elastic plate whose thickness is not so small with respect to its diameter, so the Kirchhoff's theory cannot be applied. For plate-like bodies of this kind it is convenient to apply a theory proposed by Levinson (1985). To describe mathematically the pressure distribution and the extent of the contact area we adopt the Hertz's theory. By combining these theories we derive an impact law in elastic plate-like bodies.

In this paper we analyze the low velocity impact problem of a rigid sphere against a circular elastic plate. The solution of the equations of the three dimensional theory of the elastodynamic is obtained by using a semi-inverse method and a solving technique based on the method of the separation of variables [1].

We consider a circular isotropic plate-like body of thickness *2h* and radius *b* referred to a system of cylindrical coordinates such that its origin is placed at the center of the middle plane. We extent the Levinson's problem in a dynamical framework with axisymmetric load conditions; the displacement field assumes the following form

$$u(r,z,t) = -g(z)\,\frac{d}{dr}W(r)\,e^{i\omega t}\ , \qquad w(r,z,t) = f(z)\,W(r)\,e^{i\omega t}\ , \tag{1}$$

where the function *W(r)* is the deflection of the middle surface, and *g(z)* and *f(z)* are functions determining the variations in the displacements through the thickness of the plate [2].
We assume respectively the following conditions over the mantle and on the upper and lower faces

$$w(b,z,t)=0\ ,\ \ \sigma_{rz}(r,\pm h,t)=0\ ,\ \ \sigma_{zz}(r,-h,t)=0\ \ \text{and}\ \ \sigma_{zz}(r,+h,t)=p(r,t)\ .$$

For the explicit expression of the function *p(r,t)* we adopt the periodic Hertz's normal pressure distribution

$$p(r)=\frac{3}{2}\frac{P\sqrt{a^2-r^2}}{\pi a^3}\ , \qquad a=\left(\frac{3R(2\mu+\lambda)}{16\mu(\mu+\lambda)}\right)^{\frac{1}{3}}P^{\frac{1}{3}} \tag{2}$$

where *a* is the contact area radius, *R* the radium of the rigid sphere and *P* the resultant pressure [3].
The pressure *p(r)* is written as a Fourier Bessel expansion on the interval (0,b)

$$p(r)=\sum_{j=1}^{\infty}A_j J_0\left(f_j r\right)\ , \ \text{with} \tag{3}$$

$$A_j = 3\frac{\sin\left(cP^{\frac{1}{3}}f_j\right)-cf_jP^{\frac{1}{3}}\cos\left(cP^{\frac{1}{3}}f_j\right)}{\pi b^2 c^3 f_j^3\left[J_1\left(bf_j\right)\right]^2}\ , \qquad c=\left(\frac{3R(2\mu+\lambda)}{16\mu(\mu+\lambda)}\right)^{\frac{1}{3}}$$

and $f_j = Z_j/b$ where Z_j are the *j*-th zeros of the *0*-order Bessel function of the first type. By substituting the displacement field in the linear elastodynamic equations for isotropic material, we obtain

$$\left(\mu g''-(\lambda+\mu)f'-\frac{1}{r^2}(\lambda+2\mu)g+\rho\omega^2 g\right)W' + (\lambda+2\mu)\left(\frac{1}{r}W''+W'''\right)g=0\ , \tag{4}$$

$$\left((\lambda+2\mu)f''+\rho\omega^2 f\right)W+\left(\mu f-(\lambda+\mu)g'\right)\left(\frac{1}{r}W'+W''\right)=0\ , \tag{5}$$

with ρ the density and λ and μ the Lamé moduli.

The function *W(r)* can be written as

$$W(r)=BJ_0(kr)+DY_0(kr)\ ,$$

where $J_0(kr)$ and $Y_0(kr)$ are the 0-order Bessel functions of the first and second type respectively; since the second type Bessel function is unbounded in *r=0*, the coefficient *D* is zero. The condition on the mantle yields that *k* is f_j (*j*=1,2..), so we obtain the explicit solution of the (5), in the *j-th* term, as [4]

$$W_j(r)=J_0(f_j r)\ . \tag{6}$$

Now, by substituting the (6) in the (4) and, by differentiating it with respect to *z*, we get the solutions of the function *f(z)* and *g(z)* in the form

$$f_j(z)=C_1^{(j)}\cosh(\alpha_j z)+C_2^{(j)}\sinh(\alpha_j z)+C_3^{(j)}\cosh(\beta_j z)+C_4^{(j)}\sinh(\beta_j z)$$

$$g_j(z)=-\frac{\alpha_j}{f_j^2}\left(C_1^{(j)}\sinh(\alpha_j z)+C_2^{(j)}\cosh(\alpha_j z)\right)-\frac{1}{\beta_j}\left(C_3^{(j)}\sinh(\beta_j z)+C_4^{(j)}\cosh(\beta_j z)\right). \tag{7}$$

The coefficients $C_1^{(j)}, C_2^{(j)}, C_3^{(j)}$ and $C_4^{(j)}$ are uniquely determined by the conditions on the upper and lower faces of the body.

The displacement field is obtained by considering the sum on all value of *j*, so we have the following expansions

$$u(r,z,t) = -\sum_{j=1}^{\infty}\left(\frac{a_j}{f_j}\left(C_1^{(j)}\sinh(a_j z) + C_2^{(j)}\cosh(a_j z)\right) + \frac{f_j}{b_j}\left(C_3^{(j)}\sinh(b_j z) + C_4^{(j)}\cosh(b_j z)\right) \right) J_1(f_j r) e^{iwt} , \tag{8}$$

$$w(r,z,t) = \sum_{j=1}^{\infty}\left(C_1^{(j)}\cosh(a_j z) + C_2^{(j)}\sinh(a_j z) + C_3^{(j)}\cosh(b_j z) + C_4^{(j)}\sinh(b_j z) \right) J_0(f_j r)\, e^{iwt} . \tag{9}$$

where $a_j \equiv \sqrt{f_j^2 - \frac{rw^2}{m}}$ and $b_j \equiv \sqrt{f_j^2 - \frac{rw^2}{(l+2m)}}$.

By using (9) in $z=+h$ and $r=0$ we have the *impact law* in a circular thick elastic plate in the form

$$d^D = \sum_{j=1}^{\infty} K_j^D \left(\sin\left(cP^{\frac{1}{3}} f_j \right) - cP^{\frac{1}{3}} f_j \cos\left(cP^{\frac{1}{3}} f_j \right) \right) e^{iwt} , \tag{10}$$

with

$$K_j^D = \frac{8m(m+l)\,rw^2 b_j}{(2m+l)\,p\,Rb^2 f_j^3 \left(J_1(bf_j)\right)^2} \left(\frac{1}{D_1^{(j)}}\left(1+e^{2b_j h}\right)\left(1+e^{2a_j h}\right) - \frac{1}{D_2^{(j)}}\left(1-e^{2b_j h}\right)\left(1-e^{2a_j h}\right) \right) ,$$

$$D_1^{(j)} = \left(r^2w^4 + 4f_j^2 m\left(f_j^2 m - rw^2\right)\right)\left(1+e^{2a_j h}\right)\left(1-e^{2b_j h}\right) - 4f_j^2 m^2 a_j b_j \left(1-e^{2a_j h}\right)\left(1+e^{2b_j h}\right) ,$$

$$D_2^{(j)} = \left(-r^2w^4 - 4f_j^2 m\left(f_j^2 m - rw^2\right)\right)\left(1-e^{2a_j h}\right)\left(1+e^{2b_j h}\right) + 4f_j^2 m^2 a_j b_j \left(1+e^{2a_j h}\right)\left(1-e^{2b_j h}\right) .$$

In the static case we have the following *contact law*

$$d^S = \sum_{j=1}^{\infty} K_j^S \left(\sin\left(cP^{\frac{1}{3}} f_j \right) - cP^{\frac{1}{3}} f_j \cos\left(cP^{\frac{1}{3}} f_j \right) \right) , \quad ,$$

with

$$K_j^S = \frac{8\left(1 - e^{8f_j h} - 8f_j h e^{4f_j h}\right)}{p\,Rb^2 f_j^4 \left[J_1(bf_j)\right]^2 \left(-1 + e^{4f_j h} + 4f_j h e^{2f_j h}\right)\left(1 - e^{4f_j h} + 4f_j h e^{2f_j h}\right)} .$$

By using the analytical solution (10), in fig. 1 we show the impact law in the case of low frequencies for different ratios *h/b*. As expected, we remark the agreement with the hertzian law (**d**$_{Hertz}$) for high values of the ratio *h/b*.

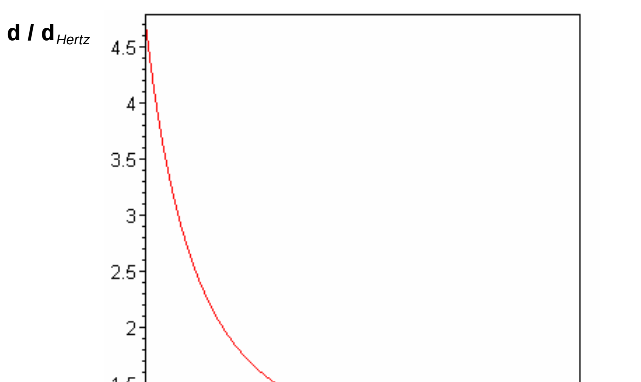

Fig. 1. Impact law for different ratios *h/b*.

References

[1] Sburlati R.: An exact solution for the impact law in thick elastic plates, *Int.J.Sol Stru*, in press 2004.
[2] Levinson M.: The simply supported rectangular plate: an exact three-dimensional linear elastic solution, *J.Elast* **7**:283-291, 1985.
[3] Stronge W.J., Impact Mechanics. Cambridge University Press, 2000.
[4] Sneddon I.N., Mixed Boundary Value Problems in Potential Theory; John Wiley &Sons Inc., 1966