The Rosetta Stone of \( L \)-functions

Enrico Bombieri

Institute for Advanced Study, School of Mathematics, Princeton, NJ 08540, USA
eb@math.ias.edu

THE ROSETTA STONE

The Rosetta Stone carries an identical text with parallel inscriptions in hieroglyphs, demotic and Greek. It is associated with the famous egyptologist Champollion (1790–1832), who used it, after he had deciphered demotic, as the starting point for reading hieroglyphs. He recognized on the Rosetta stone the name Ptolmys in Greek and demotic, and from there he identified the same name in hieroglyphs, written in a cartouche. Three years later, in 1821, while studying corresponding hieroglyphic and Greek texts on an obelisk transported to England by Giovanni Belzoni (1778–1823), he recognized the name Kliopadra, thus getting the values of twelve hieroglyphs. From there, he was able to complete the monumental task he had started in 1808 at the age of eighteen.

ZETA AND \( L \)-FUNCTIONS [13]

Number fields. The Dedekind zeta function of a number field \( K \) is defined by

\[
\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left( 1 - \frac{1}{N(\mathfrak{p})^s} \right)^{-1},
\]

where \( \mathfrak{a} \) runs over all integral ideals of \( K \), \( \mathfrak{p} \) over all prime ideals of \( K \), and where \( N(\mathfrak{a}) \) is the absolute norm from \( K \) to \( \mathbb{Q} \).

This is a generalization of \( \zeta(s) \) with similar good properties.
Theorem 1.1 (Hecke 1920).
(i) The Dedekind zeta function is meromorphic of order 1 with a simple pole at $s = 1$ with residue $2^{r_1}(2\pi)^{r_2}Rh/(w\sqrt{|\Delta(K)|})$, where $r_1$ is the number of real embeddings of $K$, $r_2$ is the number of pairs of complex embeddings of $K$, $w$ is the order of the group of roots of unity in $K$, $R$ is the regulator of $K$, $h$ is the class number and $\Delta(K)$ the discriminant of $K$.
(ii) We have a functional equation:
$$|\Delta(K)|^{s/2}\left(\pi^{-s/2}\Gamma(s/2)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_K(s)$$
which remains invariant by the change of variable $s \mapsto 1 - s$.

HECKE CHARACTERS

Absolute values. For each place $v$, let $\|x\|_v$ be the associated absolute value:
$$\|x\|_v = \begin{cases} N(p)^{-\ord_p(x)} & \text{if } v = p, \\ |x| & \text{if } v = \mathbb{R}, \\ |x|^2 & \text{if } v = \mathbb{C}. \end{cases}$$

Ideles. The idele group $J$ is $J = \prod_v K_v^\times$, with the product restricted to elements $x$ with almost every factor $\|x\|_v = 1$, together with a suitable topology. Note that $K^\times \subset J$ via the diagonal embedding.

Hecke Grössencharacter. A continuous homomorphism $\psi : K^\times \backslash J \to T$.

HECKE CHARACTERS, II

The conductor. A place $v$ is unramified for $\psi$ if
$$\psi_v(x_v) := \psi((1, ..., 1, x_v, 1, ...)) = 1$$
whenever $\|x_v\| = 1$. The conductor $f$ of $\psi$ is the ideal
$$f = \prod_{p \text{ ramified}} p^{m_v},$$
where $m_v$ is the smallest exponent for which $\psi_v(x_v) = 1$ for $x_v \in 1 + p^{m_v}$.

Grössencharacter of an ideal. It suffices to define it on prime ideals, as $\psi(p) = \psi(\varpi_v)$ with $\varpi_v$ a uniformizer of $p$ if $p \nmid f$, and 0 otherwise.
HECKE L-FUNCTIONS

For $K_v = \mathbb{R}$ or $\mathbb{C}$, we have

$$
\psi_v(x_v) = \left( \frac{x_v}{|x_v|} \right)^{m_v} |x_v|^{i\tau_v},
$$

where $m_v = 0$ or $1$ if $K_v = \mathbb{R}$ and $m_v \in \mathbb{Z}$ if $K_v = \mathbb{C}$.

**Theorem 1.2 (Hecke 1920).**

(i) The $L$-function

$$
L(s, \psi) = \sum_a \frac{\psi(a)}{N(a)^s} = \prod_{p \text{ unramified}} \left( 1 - \frac{\psi(p)}{N(p)^s} \right)^{-1}
$$

attached to a non-trivial Grössencharacter $\psi$ is entire of order $1$.

(ii) Let $\Delta_\psi = |\Delta(K)| N(f_\psi)$ and define

$$
A(s, \psi) = (\Delta_\psi)^{-s/2} \prod_{K_v = \mathbb{R}} \pi^{-s/2} \Gamma \left( \frac{s}{2} + c_v \right) \prod_{K_v = \mathbb{C}} (2\pi)^{-s} \Gamma(s + c_v) L(s, \psi),
$$

where $c_v = (i\tau_v + |m_v|)/2$. Then

$$
A(s, \psi) = w(\psi) A(1 - s, \overline{\psi}),
$$

for some complex number $w(\psi)$ with $|w(\psi)| = 1$.

ARTIN L-FUNCTIONS

$L/K$ is a Galois extension of degree $d$ of $K$ with Galois group $G$. Let $\pi : G \to GL_n(\mathbb{C})$ be a representation of $G$.

The lift $\mathfrak{p} \mathcal{O}_L$ to $L$ of a prime ideal $\mathfrak{p}$ of $K$ factors as

$$
\mathfrak{p} \mathcal{O}_L = \prod_{i=1}^r \mathfrak{p}_i^e.
$$

e is the ramification index, $|\mathcal{O}_L/\mathfrak{p}_i| = |\mathcal{O}_K/\mathfrak{p}|^f$ and $efr = d$.

$G$ acts on $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$ by permutations; the subgroup $G_{\mathfrak{P}}$ fixing $\mathfrak{P}$ is the decomposition group of $\mathfrak{P}$. The elements $\sigma \in G_{\mathfrak{P}}$ with

$$
\sigma x \equiv x^{N(\mathfrak{p})} \quad (\text{mod } \mathfrak{P})
$$

form a right and left coset $(\mathfrak{P}, L/K)$ (the Frobenius substitution) of the inertia group $I_{\mathfrak{P}}$ fixing $K(\mathfrak{P}) = \mathcal{O}_L/\mathfrak{P}$. Changing $\mathfrak{P}$ into $\eta \mathfrak{P}$ yields $(\eta \mathfrak{P}, L/K) = \eta(\mathfrak{P}, L/K)\eta^{-1}$. 
ARTIN L-FUNCTIONS, II [13]

Let $\chi = \text{Tr}(\pi)$ be the character of $\pi$ and define for $\sigma \in (\mathfrak{P}, L/K)$:

$$
\chi(p^m) = |I_p|^{-1} \sum_{\tau \in I_p} \chi(\sigma^m \tau).
$$

Then $\chi(p^m)$ is independent of the choices of $\mathfrak{P}$ and $\sigma$.

The local $L$-function.

$$
L_p(s, \chi, L/K) := \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \chi(p^m) N(p)^{-ms} \right).
$$

The Artin $L$-function.

$$
L(s, \chi, L/K) = \prod_p L_p(s, \chi, L/K).
$$

THE ARTIN FORMALISM [13]

$L(s, 1, L/K) = \zeta(s, K)$.

Direct sum.

$$
L(s, \chi_1 + \chi_2, L/K) = L(s, \chi_1, L/K) L(s, \chi_2, L/K).
$$

Restriction. Let $L' \supset L \supset K$ be Galois extensions. Then

$$
L(s, \chi, L'/K) = L(s, \chi, L/K).
$$

Induction. Let $L' \supset L \supset K$ be Galois extensions and let $\chi^*$ be the character of $G = \text{Gal}(L'/K)$ induced by a character $\chi$ of $H = \text{Gal}(L'/L)$. Then

$$
L(s, \chi, L'/L) = L(s, \chi^*, L'/K).
$$

The character $\chi^*$ is the unique character such that $(\chi^*, \psi)_G = (\chi, \psi|_H)_H$ where $(\ , \ )_G$ is the scalar product on central functions on $G$ normalized with $(1, 1)_G = 1$.

ARTIN L-FUNCTIONS, III

Theorem 1.3 (Artin 1923). We have

$$
L(s, \chi, L/K) = \prod_p \det \left[ I - \frac{\pi(p)}{N(p)^s} \right]^{-1},
$$

where

$$
\pi(p) = \pi(\sigma)|I_p|^{-1} \sum_{\tau \in I_p} \pi(\tau).
$$

For the proof, note that $|I_p|^{-1} \sum_{\tau \in I_p} \pi(\tau)$ is an idempotent.
AN EXAMPLE

\[ G = \{\pm 1\}, \quad L/K = \mathbb{Q}(\sqrt{D})/\mathbb{Q}, \quad \text{D squarefree}. \quad \Delta = D \text{ if } D \equiv 1 \pmod{4}, \text{ otherwise } \Delta = 4D. \]

Case 1: \( p \nmid \Delta, \ (p) = p \mathfrak{p} \). Here \( G_p = \{1\} \) and \((p, G) = (\mathfrak{p}, G) = 1\) by Fermat’s Little Theorem.

Case 2: \( p \nmid \Delta, \ (p) \text{ prime.} \) Here \( G_p = \{\pm 1\}, \ I_p = \{1\} \) and \((p, G) = -1, \) because if \((p)\) does not split then \( D\) is not a quadratic residue \( \pmod{p}\).

Case 3: \( p \mid \Delta, \ (p) = p^2 \). Here \( G_p = \{\pm 1\}, \ I_p = \{\pm 1\} \) and \((p, G) = \{\pm 1\} \).

AN EXAMPLE, II

Take \( \pi \) to be the regular representation

\[ \pi\{1, -1\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}. \]

The local factors are:

\[ \det \begin{bmatrix} 1 - p^{-s} & 0 \\ 0 & 1 - p^{-s} \end{bmatrix} = (1 - p^{-s})^2 \quad \text{if } p \text{ splits;} \]
\[ \det \begin{bmatrix} 1 & -p^{-s} \\ -p^{-s} & 1 \end{bmatrix} = 1 - p^{-2s} \quad \text{if } p \text{ remains prime;} \]
\[ \det \begin{bmatrix} 1 - \frac{1}{2}p^{-s} & -\frac{1}{2}p^{-s} \\ -\frac{1}{2}p^{-s} & 1 - \frac{1}{2}p^{-s} \end{bmatrix} = 1 - p^{-s} \quad \text{if } p \text{ ramifies.} \]

Then \( L(s, \pi, L/K) = \zeta(s)L(s, (\Delta)). \)

AN EXAMPLE, III

Identifying \( L(s, \pi, L/K) = \zeta(s)L(s, (\Delta)) \) (where \((\Delta)\) is the Kronecker symbol) with \( \zeta(s, L/K) = \zeta(s)L(s, \chi_\Delta) \), where now \( \chi_\Delta \) is the Dirichlet character, is the quadratic reciprocity law. In the general case of abelian extensions \( L/K \), the corresponding result is Artin’s reciprocity law of class field theory. For example, it implies as a special case the famous Kronecker–Weber theorem that every abelian extension of \( \mathbb{Q} \) is a subfield of a cyclotomic field.

ARTIN L-FUNCTIONS, IV

Artin’s Conjecture. The L-function \( L(s, \chi_\pi, L/K) \) associated to a non-trivial irreducible representation \( \pi \) of Gal\( (L/K) \) is an entire function of \( s \) of order 1.
Known only in special cases. For $G$ abelian, Artin. Also known for characters expressible as linear combinations with positive coefficients of characters induced by cyclic subgroups of $G$. For $\dim(\pi) = 2$ and $G = S_3$ and $G = S_4$, Langlands [13], Tunnell [22]. For $G = A_5$ with some conditions, Taylor [20], Shepherd-Barron & Taylor [18], Buzzard & Dickinson & Shepherd-Barron & Taylor [2], Buzzard & Stein [3].

**Brauer’s Theorem.** $L(s, \chi, L/K)$ is meromorphic of order 1 and satisfies a functional equation ($\Lambda = L \times \{\Gamma\text{-factors}\}$)

$$
\Lambda(s, \chi, L/K) = w(\chi, L/K)\Lambda(1-s, \bar{\chi}, L/K),
$$

where $\bar{\chi}$ is the character of the contragredient of $\pi$.

**SOME MODULAR FORMS**

Let $f(z)$ be a holomorphic modular form of weight $k$ for a subgroup $\Gamma < \Gamma(1)$ generated by $z \mapsto z + h$ and $z \mapsto -1/z$:

$$
f(\gamma z) = \varepsilon(\gamma)(cz + d)^k f(z), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
$$

with $\varepsilon(\gamma)$ an appropriate set of multipliers (Nebentypus).

Then $f(z)$ is periodic and has a Fourier expansion

$$
f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z / h}.
$$

By the Mellin transform, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$ is meromorphic of order 1, with a simple pole at $s = k$ if $a_0 \neq 0$, and

$$(2\pi/h)^{-s} \Gamma(s) L(s, f) = w (2\pi/h)^{s-k} \Gamma(k-s) L(k-s, f),$$

for $w = i^k \varepsilon([z \mapsto -1/z])$.

**A CONVERSE THEOREM**

**Theorem 1.4 (Hecke 1936).** Conversely, given $L(s, f)$ with the above properties one recovers a modular form $f$ of weight $k$ for $\Gamma$, by setting $a_0 = i^{-k} w$ in case $L(s, f)$ has a pole at $s = k$.

**Example.** Take $h = 2$ and

$$
\theta(z) = \frac{1}{2} \sum_{n=\infty}^{\infty} e^{\pi i n^2 z}.
$$

Then $\theta(z)$ is a modular form of weight $1/2$, multiplier 1 and $L(s, \theta) = \zeta(2s)$. The space of such forms for $\Gamma$ has dimension 1, hence $\zeta(2s)$ is the unique (up to a scalar) Dirichlet series satisfying the same functional equation as $\zeta(2s)$. 
A CONVERSE THEOREM, II

Hecke’s converse result uses only the cusp at $i\infty$. Weil’s new idea: Control of the Fourier expansions at every cusp by ‘twisting’ the Dirichlet series with Dirichlet characters.

**Theorem 1.5 (Weil 1967).** Let $L(s, f) = \sum a_n n^{-s}$ and write, for a Dirichlet character $\chi \pmod{r}$:

$$A(s, f \otimes \chi) = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n \chi(n)n^{-s}.$$ 

Suppose there are $N, k$ such that for every primitive $\chi \pmod{r}$ with $(r, N) = 1$ we have that $A(s, f \otimes \chi)$ is entire of order 1 and

$$A(s, f \otimes \chi) = w_\chi r^{-1}(r^2 N)^{k/2-s} \Lambda(k - s, f \otimes \chi)$$

with $w_\chi = i^k \chi(N)G(\chi)^2$ and $G(\chi) = \sum_n \chi(n) \exp(2\pi i n/r)$.

Then $L(s) = L(s, f)$ with $f$ a holomorphic cusp form for $\Gamma_0(N)$.

THE HASSE–WEIL ZETA FUNCTION

**The Hasse–Weil zeta function.** Let $V/K$ be a variety over a number field. Then for all except finitely many prime ideals $p$ the reduction $V_p$ is defined over the finite field $\mathcal{O}_K/p$ and yields a Zeta function $Z(T, V_p)$. The global zeta function of $V$ is now

$$\prod_p Z(N(p)^{-s}, V_p),$$

where the product runs over all prime ideal where the reduction is defined.

If $V'/K$ is another model of $V$ the zeta functions are the same up to finitely many factors.

**A difficulty.** Define ‘good factors’ even at places of bad reduction and ‘good models’ for which the zeta function behaves nicely.

**SOME EXAMPLES**

**Example.** Let $V$ be the projective space $\mathbb{P}^n/\mathcal{O}_K$. Then

$$\zeta(s, V) = \zeta_K(s)\zeta_K(s-1)\cdots\zeta_K(s-n).$$

**Example (Deuring, 1953–57).** Let $E$ be an elliptic curve over a number field $L$, with complex multiplication in the imaginary quadratic field $K = \text{End}(E) \otimes \mathbb{Q}$. Assume $K \subset L$. Then there is a model of $E/L$ and a Hecke Grössencharacter $\psi$ of $L$ such that

$$\zeta(s, E) = \zeta_L(s)\zeta_L(s-1)L(s-1/2, \psi)L(s-1/2, \bar{\psi}).$$

**Example (Taniyama, 1957).** The previous example extends to abelian varieties of CM type.
THE TANIYAMA CONJECTURE

Eichler (1953) proved the functional equation for zeta functions of modular curves \( X_0(N) = \mathcal{H}/\Gamma_0(N) \) of genus 1, by showing that in this case \( L(s, X_0(N)) \) was the Mellin transform of a cusp form of weight 2. Shimura extended this to certain other cases. On the basis of this evidence, Taniyama suggested this held in general.

**Conjecture 1.6 (Taniyama 1955, Shimura, Weil, . . .).** Every elliptic curve over \( \mathbb{Q} \) is uniformized by a cusp form of weight 2 for \( \Gamma_0(N) \). Equivalently, every elliptic curve \( E/\mathbb{Q} \) admits the modular curve \( X_0(N) \) as a ramified covering, for some \( N \).

Proved by Wiles, Wiles & Taylor, in the semistable case (namely multiplicative bad reduction only) and now in general by Breuil & Conrad & Diamond & Taylor.

FIBONACCI AND CONGRUENT NUMBERS

**Fibonacci sequence**

\[ f_{n+1} = f_n + f_{n-1} \]

The *Liber Quadratorum*

**Fibonacci’s formula**

\[ (a^2 + b^2)(c^2 + d^2) = (ad \pm bc)^2 + (ac \mp bd)^2 \]

**Congruent numbers** \( m \)

\[ h^2 + m = \-box{} \quad h^2 - m = \-box{} \]

\[ h^2 = a^2 + b^2 \quad m = 2ab = 4 \times \text{area} \]

Fibonacci: \( m \neq \box{} \)

**The congruent number 157**

The smallest solution is (Zagier)

\[ a = 340164924191321525608770 \]
\[ 411340519227716149933203 \]

\[ b = 41134051927716149933203 \]
\[ 433331133874295226192207 \]

**CONGRUENT NUMBERS, II**

The solution? (Tunnell 1983). Let \( g = q \prod (1 - q^8) (1 - q^{16}) \), \( \theta_2 = \sum q^{2n^2}, \theta_4 = \sum q^{4n^2}, q = e^{\pi i z} \), and define cusp forms of weight 3/2 for \( \Gamma_0(32) \):

\[ g\theta_2 = \sum a(n)q^n, \quad g\theta_4 = \sum b(n)q^n. \]
Then a squarefree \( m \) is not a congruent number if \( a(m) \neq 0 \) (\( m \) odd), and if \( b(m/2) \neq 0 \) (\( m \) even). The converse is true if the B & S-D conjecture holds for \( g^2 = x^3 - m^2 x \) (or \( my^2 = x^3 - x \)).

Comments on proof: \( m \) is congruent if and only if the curve \( E_m \) defined by \( my^2 = x^3 - x \) has rational solutions with \( y \neq 0 \). The connection with cusp forms uses a deep result of Waldspurger for computing \( L(1, E_m) \) (for example, for odd \( m \) it yields \( L(1, E_m) = \beta a(m)^2/(4\sqrt{m}) \), \( \beta = \int_1^\infty (x^3 - x)^{-1/3} dx = 2.62205 \ldots \)), theta lifts and the B&S-D conjecture.

THE NEW ROSETTA STONE

The intertwining of geometry, automorphic theory, and \( L \)-functions, was compared by André Weil with the reading of a Rosetta stone for mathematics.

**Motivic writing.** \( L \)-functions can be defined from geometry (Hasse, Weil, \( \ldots \))

**Galois writing.** \( L \)-functions can be defined from finite dimensional representations of the absolute Galois group \( \text{Gal}(\overline{K}/K) \) acting on a vector space \( V \) (Artin, Weil, Serre, \( \ldots \))

**Automorphic writing.** \( L \)-functions can be defined globally from automorphic forms on algebraic groups modulo discrete subgroups (Hecke, Langlands, \( \ldots \))

AUTOMORPHIC \( L \)-FUNCTIONS [16]

Langlands \( \sim 1970 \). Vastly extended the concept of automorphic form and automorphic \( L \)-function.

**Remark.** A Hecke character is nothing else than a representation of \( GL_1(\mathbb{A}) \) in the space of continuous functions on \( GL_1(K) \setminus GL_1(\mathbb{A}) \).

An automorphic representation is an irreducible component \( \pi \) of a representation of the group \( GL_n(\mathbb{A}) \) on the space of continuous functions on \( GL_n(K) \setminus GL_n(\mathbb{A}) \) (with technical conditions). One can then attach to \( \pi \) an \( L \)-function, with functional equation \( L(s, \pi) = w(\pi)L(1-s, \tilde{\pi}) \) with \( \tilde{\pi} \) the contragredient. Also \( \pi \) decomposes as a tensor product \( \otimes \pi_v \) of local components, yielding an Euler product for \( L(s, \pi) \) with standard factors of degree \( n \).

AUTOMORPHIC \( L \)-FUNCTIONS, II [16]

Automorphic \( L \)-functions (Langlands [14]). A concept which fuses together the Artin and Hecke concepts. Given a connected reductive group \( G/K \) and a finite extension \( L/K \), one considers an extension \( L^G \) of \( G \) by \( \text{Gal}(L/K) \),
a finite dimensional complex representation $\rho$ of $L^G$, and a representation $\pi$ of $G(\mathbb{A})$. The theory of Hecke operators gives us, for each local factor $\pi_v$ of $\pi$, a conjugacy class $g_v = g(\pi_v) \in L^G$ which generalizes the notion of Frobenius substitution.

The automorphic $L$-function associated to $\rho$ and $\pi$ is a product of local factors where for almost every $v$ one has
\[
L_v(s, \pi_v, \rho) = \det \left( I - \rho(g_v)N(p)^{-s} \right)^{-1}.
\]

AUTOMORPHIC $L$-FUNCTIONS, III [16]

The goal. The reciprocity law, namely: Given $\rho$ and $\pi$ there is $\pi'$ of $G(\mathcal{O})$ such that $L_v(s, \pi_v, \rho) = L_v(s, \pi'_v)$ for almost every $v$ and $L(s, \pi, \rho) = L(s, \pi')$.

The first tool. The principle of functoriality. If $H$ and $G$ are two reductive groups and $L^G \rightarrow \text{Gal}(L/K)$ factors as
\[
L^G \xrightarrow{\phi} L^H \rightarrow \text{Gal}(L/K),
\]
then one expects for each $\pi$ for $G$ to attach $\Pi$ for $H$ such that one has the equality of conjugacy classes \{g(\Pi_v)\} = \{\phi(g(\pi_v))\} for almost every $v$. An important special case is base change [15], known in some cases.

Others tools. Converse theorems (Cogdell and Piatetski-Shapiro [4]), theta-liftings, the Selberg–Arthur trace formula.

READING THE ROSETTA STONE

Automorphic = Galois. This is a key step with very deep implications. For $GL_1$, it is Artin’s abelian reciprocity law of class field theory, a vast generalization of the quadratic reciprocity law. For $GL_2$ it is known for dihedral (reduces to $GL_1$), tetrahedral and octahedral representations and many (but not yet all) icosahedral representations, i.e. $G = S_3, S_4, A_5$ ([15], [23], [20], [18], [2], [3]).

Galois = Motivic. This is also a key step, understood in very few cases, namely $GL_1$ (Artin, Hecke), elliptic curves $E/\mathbb{Q}$ (the Taniyama conjecture).

Motivic = Automorphic. Known in very few cases. Possibly there are more automorphic $L$-functions than motivic $L$-functions coming from geometry.
APPLICATIONS

If
\[ L(s, f) = \prod_v \left( 1 - \omega_1(p)N(p)^{-s} \right)^{-1} \left( 1 - \omega_2(p)N(p)^{-s} \right)^{-1} \]
the symmetric \( k \)-th power is
\[ L(s, \text{Sym}^k f) = \prod_v \prod_{j=0}^k \left( 1 - \omega_1(p)^j \omega_2(p)^{k-j} N(p)^{-s} \right)^{-1}. \]

A deep conjecture is that these symmetric powers have analytic continuation and functional equation. A success of these methods has been to prove such a conjecture for \( k = 2, 3, 4 \) (\( k = 2 \) by Rankin and Selberg for classical cusp forms, Gelbart and Jacquet [7] in general, \( k = 3 \) and 4 by Kim and Shahidi [11]).

CLASSICAL PROBLEMS

We have a good formal understanding for \( L \)-functions for Dirichlet characters and cusp forms for congruence subgroups of \( \Gamma(1) \), but only limited information on analytical behavior. The functional equation has the form \( \Lambda(s, f) = w(f)\Lambda(k-s, f) \), \( k = 1 \) or 2. An Euler product for \( L(s, f) \) has local factors given by polynomials of degree \( k \). The integer \( k \) is the degree of \( L(s, f) \).

The Generalized Riemann Hypothesis (GRH). The zeros of the functions \( \Lambda(s, f) \) with Euler product all lie on the vertical line at the center \( k/2 \) of the critical strip (the critical line).

The Generalized Lindelöf Hypothesis (GLH). For every fixed \( \varepsilon > 0 \), \( L(s, f) \) has order \( N^\varepsilon(|s| + 1)^{\varepsilon} \) in the half-plane to the right of the critical line (except at a possible pole). Here \( N \) is the conductor.

CLASSICAL PROBLEMS, II

Let \( \mu(\sigma, f) \) be defined as the best exponent for which
\[ |L(\sigma + it, f)| \ll (N(|t| + 1))^{\mu(\sigma, f) + \varepsilon}. \]

Then the functional equation and convexity yields \( \mu(\sigma, f) \leq (k - \sigma)/2 \) for \( 0 \leq \sigma \leq k \). The Lindelöf hypothesis is \( \mu(\sigma, f) = 0 \) for \( k/2 \leq \sigma \leq k \); a subconvexity bound is the statement \( \mu(\sigma) < (k - \sigma)/2 \) for \( k/2 \leq \sigma \leq k \). Such a statement has deep arithmetic consequences which are unattainable using the convexity bound alone. The estimate in the conductor is particularly hard (Duke & Friedlander & Iwaniec [5], [6]).
CLASSICAL PROBLEMS, III

The Generalized Ramanujan Conjecture (GRC). Ramanujan conjectured, on the basis of numerical evidence, that $|\tau(p)| \leq 2p^{11/2}$ for every prime $p$. In general, this is about the coefficients $a_n$ in $L(s,f) = \sum a_n n^{-s}$. Then GRC is the statement

$$|a_n| \ll n^{(k-1)/2+\varepsilon},$$

for every fixed $\varepsilon > 0$.

GRC is a deep statement. For $K = \mathbb{Q}$ and $k = 2$, it is known when $f$ is a holomorphic cusp form of even integral weight for $\Gamma(1)$, as a consequence of Deligne’s GRH for varieties over finite fields (1974). It is open already for $f$ a non-holomorphic cusp form (Maaß wave form) for $\Gamma(1)$.

FAMILIES

Let $\{\lambda_j\}$ be a real sequence, $\lambda_j \sim j$. Consider the gaps $\Delta_j = \lambda_{j+1} - \lambda_j$ and set

$$\mu_1(N)[a,b] = \frac{1}{N} \# \{ j : \Delta_j \in [a,b] \}, \quad 0 \leq j < N.$$

Quite often one finds a Cauchy–Poisson distribution

$$\mu_1(N) \rightarrow e^{-x}dx.$$

Example. This is expected for the sequence $\{p_j/\log p_j\}$, although it remains hopeless to prove.

Example. Again, expected for the (ordered) sequence $\{\pi^2/4 + \pi^2/4 m^2 + \sqrt{2n^2}\}$ and, again, completely open.

Problem. What is the expected behavior for $\{1/(2\pi) \gamma \log \gamma\}$, where $\zeta(1/2 + i\gamma) = 0$ and $\gamma > 0$? (There are applications.)

FAMILIES, II [9]

For $A \in U(N)$, consider the eigenvalues $z_1, \ldots, z_N$ on the unit circle ordered by increasing argument (mod $2\pi$). Let

$$\mu_{k,N}(A)[a,b] = \frac{1}{N} \# \left\{ j : \frac{N}{2\pi} \arg(z_j + k/z_j) \in [a,b] \right\}.$$

Theorem 1.7 (Gaudin 1961, Katz & Sarnak 1999).

$$\int_{U(N)} \mu_{k,N}(A) dA \rightarrow \mu_k([a,b]),$$

(Here $dA$ is the normalized Haar measure) with
\[ d\mu_k = \frac{d^2}{ds^2} \left( \sum_{j=0}^{k-1} \frac{k-j}{j!} \left( \frac{\partial}{\partial T} \right)^j \det (I + TK(s)) \right|_{T=1} \right) ds \]

and \( K(s) \) the operator defined by

\[ K(s)\phi(x) = \int_{-s/2}^{s/2} \frac{\sin(x-y)}{\pi(x-y)} \phi(y) dy. \]

**Conjecture 1.8 (Montgomery, Katz & Sarnak).** The sequence of zeros of \( \zeta(s) \) satisfies the above \( U(N) \) statistics for eigenvalues of random unitary matrices.

**FAMILIES, III**

Extensive calculation by Odlyzko \[17\] with zeros of \( \zeta(s) \) around \( 10^{20} \) show a total agreement with the prediction. The only problem is: Why is this so?

There are similar formulas for the other classical groups \( SU(N), O(N), SO(N), USp(N) \). The amazing thing is that certain families of \( L \)-functions seem to follow the same correlations. For example, the distribution of the \( j \)-th zero of \( L(s, \chi) \), \( \chi \) a primitive quadratic character, follows the \( USp \) prediction. Instead, the \( j \)-th zero of \( L(s, E \otimes \chi) \), \( E \) an elliptic curve, follows the \( O \) prediction (two cases, according to the sign \( \pm \) in the functional equation, corresponding to the two connected components \( O^\pm \) of \( O \)). There is numerical evidence that these are the true statistics and some consequences of them, pertaining to the behaviour of moments of small order of \( L \)-functions on the critical line, have also been verified unconditionally.

**FAMILIES, IV**

Katz and Sarnak have shown that in the function field analogue the prediction are verified for families for which Deligne’s theory applies, and the associated groups are nothing else than the monodromy groups of the families.

**Question.** For families of classical \( L \)-functions, what should replace monodromy so as to explain how these laws arise?

**Question.** Is this phenomenon peculiar to \( L \)-functions or is it instead the expression of a ‘universality law’ which holds in a much wider context?

**Question.** Is there a way of formulating a program to prove GRH following Deligne’s blueprint in this new context?
FAMILIES, V

The predictions on correlations have changed our way of thinking about \( L \)-functions.

Example. Until very recently, most analytic number theorists thought that the maximum order of magnitude of \( \log^+ |\zeta(1/2 + it)| \) was \( \sqrt{\log |t| \log \log |t|} \), based on the known Gaussian behavior of \( \log |\zeta(1/2 + it)| / \sqrt{\pi \log |t|} \) and a probabilistic extrapolation. On RH, it was known that this maximum order cannot exceed \( \log |t| / \log \log |t| \), but this is only an upper bound and the gap between the two was ascribed to an intrinsic ‘weakness’ of analytic methods.

Today, the work of Keating and Snaith \cite{10} leads us to believe that the true maximum order of magnitude of \( \log^+ |\zeta(1/2 + it)| \) is \( \log |t| / \log \log |t| \). This came as a total surprise to experts.

FAMILIES, VI

In the function field case, the discovery by Ulmer \cite{24} of elliptic curves over \( \mathbb{F}_p \) with rank as big as \( \log N / \log \log N \) (\( N \) is the conductor) confirms, via a Birch & Swinnerton-Dyer conjecture, the new prediction that the maximum order of magnitude of \( \log^+ |\zeta(1/2 + it)| \) is \( O(\log |t| / \log \log |t|) \).

Conclusion. The analogy with the function field case is often a good predictor for the classical case too.

Conclusion. The recent proof by Lafforgue \cite{12} in the function field case that motivic = automorphic gives support to the hoped deciphering of the new Rosetta stone, namely that motivic = automorphic also holds in the classical case.

References

The Rosetta Stone of $L$-functions