Dynamics and Output Momentum Spectrum of Electrons Under Harmonic Resonance in Gyrotron Resonators

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Abstract. Electron dynamics in gyrotron resonators are analyzed in the context of the Canonical Perturbation Method. Analytical approximate invariants of the electron motion are calculated and utilized in order to provide the phase space structure, the output momentum spectrum, and the distribution function of the electrons. The general case of electron interaction with multiple rf modes at different harmonics of the cyclotron frequency is considered.

Keywords: Microwave sources, gyrotrons, wave-particle interactions, Hamiltonian systems, canonical perturbation theory.

INTRODUCTION

The study of complex electron dynamics under interaction with single or multiple rf modes in a gyrotron resonator is a key issue for the effective design and implementation of gyrotrons. The complexity of electron motion is related to several interesting features of gyrotron operation, such as chaotic rf oscillations [1], complex output momenta spectra [2], and hysteresis-like effects [3]. In this paper, we study electron dynamics in gyrotrons in the context of the Hamiltonian formalism and the Canonical Perturbation Method (CPM) [4].

THE MODEL

The electron motion under interaction with a set of rf modes in a gyrotron resonator is governed by the equation [5]

\[
\frac{dp}{d\zeta} + ip|p|^2 = i \sum_s n_s F_s^* (p^*)^{n_s - 1} f_s(\zeta) e^{-in_s \int_0^\zeta \delta_s(\zeta)d\zeta'}
\] (1)

where \( p \) is the dimensionless transverse momentum of the electron; \( \zeta = \left( \frac{\beta z \omega_0}{2p_{0\zeta}} \right) \) \( z \) is the dimensionless coordinate; \( |F_s| \) is the normalized amplitude the \( s \)-th mode; \( \arg(F_s) \) is an initial phase difference; \( f_s \) describes the axial structure of the \( s \)-th mode; \( \delta_s(\zeta) \).
$1 - \Delta_s(\zeta); \Delta_s(\zeta) = (2/\beta_\perp^2)\frac{\partial}{\partial \omega_0}n_\omega\omega_0\ $ is the frequency mismatch between the $s$-th mode and the $n_s$-th harmonic of the cyclotron frequency; $\omega_0 = (e/m)B_0/\gamma_{rel}$ is the electron cyclotron frequency; $\omega_{c0} = (e/m)B_0/\gamma_{rel}$ is the electron cyclotron frequency at the entrance of the cavity; $\beta_\perp = v_\perp/c$ and $\beta_\parallel = v_\parallel/c$ are the normalized transverse and parallel velocities of the electron at the entrance to the cavity, respectively; $B$ is the magnetic field; $\gamma_{rel} = 1 + (e/mc^2)U$ is the relativistic factor; and $U$ is the accelerating voltage.

By transforming to action-angle variables: $p = \sqrt{2J}e^{i\theta}$, we obtain

\[
\frac{dJ}{d\zeta} = -\sum_s n_s F_s(2J)\bar{\omega}_s \text{Im}\left\{e^{in_s\theta}f_s(\zeta)e^{-in_s\omega_c\delta_s(\zeta)\zeta'}\right\}
\]

(2)

\[
\frac{d\theta}{d\zeta} = 2J - \sum_s n_s F_s(2J)\bar{\omega}_s^{-1} \text{Re}\left\{e^{in_s\theta}f_s(\zeta)e^{-in_s\omega_c\delta_s(\zeta)\zeta'}\right\}
\]

(3)

The system has the following Hamiltonian:

\[
H(J, \theta, \zeta) = H_0(J) + H_1(J, \theta, \zeta)
\]

(5)

\[
H_0(J) = J^2
\]

(6)

\[
H_1(J, \theta, \zeta) = -\sum_s F_s(2J)\bar{\omega}_s \text{Re}\left\{e^{in_s\theta}g_s(\zeta)\right\}
\]

(7)

where $g_s(\zeta) = f_s(\zeta)e^{-in_s\omega_c\delta_s(\zeta)\zeta'}$.

**CANONICAL PERTURBATION METHOD**

According to the Canonical Perturbation Method [6], we consider the system $(H)$ as a perturbation of system describing the free electron motion $(H_0)$. We seek a canonical transformation to new variables $(\bar{J}, \bar{\theta})$ described by a generating function $S$, so that the transformed Hamiltonian $\tilde{H}$ is a function of the action $\bar{J}$ alone. The first-order generating function is given by the following equation

\[
\frac{\partial S_1}{\partial \zeta} + \omega_\theta \frac{\partial S_1}{\partial \theta} = -H_1
\]

(8)

where $\omega_\theta = \partial H_0/\partial J = 2J$. The solution of (8) is [4]

\[
S_1 = \sum_s \text{Re}\left\{F_s(2J)\bar{\omega}_s e^{in_s\theta} \int_{\infty}^{\zeta} g_s(t)e^{in_s\omega_\theta(t-\zeta)} dt\right\}
\]

(9)

The new action $\bar{J}$

\[
\bar{J} = J - \frac{\partial S_1(J, \theta, \zeta)}{\partial \theta}
\]

(10)

is an approximate invariant of the motion, and the contour plots defined by the equation

\[
\bar{J}(J, \theta = \theta_0, \zeta) = \text{const}.
\]

(11)
represent the Poincaré surfaces of a section of the electron phase space.

For the case of an electron beam where all electrons have an initial action value $J_0$ with the angular values uniformly distributed over the interval $[0, 2\pi]$, the extreme values of electron action at a specific $\zeta_0$ are given by the solution of equation

$$J_0 = J \pm \max_{\theta \in [0, 2\pi]} \left\{ \left| \frac{\partial S_1(J, \theta, \zeta_0)}{\partial \theta} \right| \right\}$$ (12)

with respect to $J$.

The approximate invariants of the electron motion (10) can also be utilized to provide approximate solutions of the Vlasov equation governing the evolution of the electron distribution function. According to the method of characteristics, an approximate solution of the Vlasov equation is given by $F(J, \theta, \zeta) = F(\bar{J})$. By combining the Taylor expansions of this expression and (10), the approximate solution can be written in the following form

$$F(J, \theta, \zeta) = F_0(J) - \varepsilon \frac{\partial F_0}{\partial J} \left( \frac{\partial S_1}{\partial \theta} \right) + \varepsilon^2 \frac{1}{2} \frac{\partial}{\partial J} \left( \left( \frac{\partial S_1}{\partial \theta} \right)^2 \frac{\partial F_0}{\partial J} \right) + O(\varepsilon^3)$$ (13)

where $F_0(J)$ is uniform in $\theta$ initial distribution (at $\zeta = -\infty$). The approximate distribution function describes the collective dynamical behavior of the beam and can be used for gyrotron efficiency ($\eta_{\perp}$) calculations:

$$\eta_{\perp} \equiv \langle |p(\zeta_{\text{in}})|^2 - |p(\zeta_{\text{out}})|^2 \rangle_{\theta}$$

$$= 2 \langle J(\zeta_{\text{in}}) - J(\zeta_{\text{out}}) \rangle_{\theta}$$

$$= - \int_{-\infty}^{\infty} F_0(J) \frac{\partial}{\partial J} \left\langle \left( \frac{\partial S_1}{\partial \theta} \right)^2 \right\rangle_{\theta} dJ$$ (14)

RESULTS AND DISCUSSION

The analytical results of the previous section are applied for the cases of electron interaction with one or more rf modes having a Gaussian profile of the form $f(\zeta) = \exp\left(-\left(\frac{2\zeta}{\mu} - \sqrt{3}\right)^2\right)$, where $\mu = \pi \left( \beta_{\perp 0}^2 / \beta_{\parallel 0}^2 \right) L_G / \lambda_c$ is the dimensionless length of the resonator, with $L_G$ being the length of the resonator midsection. The frequency mismatch of each mode is $\delta_s(\zeta) = \delta_s^{(0)} + 2\delta_s^{(1)} \zeta$.

In Figs. 1-4, numerically and analytically obtained Poincaré surfaces of sections are shown for cases of electron interaction with a single rf mode at the first ($n = 1$) and the second ($n = 2$) harmonic. The approximate invariant of the motion (10) can represent the electron phase space which has an inhomogeneous structure. The extreme values of electron output momenta spectra, as calculated by (12) are shown in Fig. 5 for the case of constant frequency mismatch, while the case of a linearly varying frequency mismatch is shown in Fig. 6. Efficiency calculations based on (14) are depicted in Fig. 7 for the cases of constant (Fig. 7a) and varying (Fig. 7b) frequency mismatches for a Dirac initial
distribution function $F_0 = \delta(J - J_0)$. The realistic case of a Gaussian initial distribution function with a standard deviation $J_s$ is shown in Fig. 7c.

When electrons interact with more than one rf mode, the phase space has a resonant band structure with each resonant area corresponding to an rf mode, as shown in Fig. 8 and Fig. 9, for the cases of two modes being in resonance at a different or the same harmonic, respectively. In general, the resonant areas may be well-separated, weakly or strongly overlapping.

**ACKNOWLEDGMENTS**

This work was supported in part by the European Fusion Programme (EURATOM) and the Greek General Secretariat of Research and Technology. The sponsors do not bear any responsibility for the contents in this work.

**FIGURE 1.** (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section $\Sigma_{\theta_0=0}$. Parameter set: $F = 0.01$, $\Delta = 0.5$, $\mu = 15$, $n = 1$.

**FIGURE 2.** (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section $\Sigma_{\theta_0=0}$. Parameter set: $F = 0.01$, $\Delta = 0.5$, $\mu = 15$, $n = 2$.

**FIGURE 3.** (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section $\Sigma_{\theta_0=0}$. Parameter set: $F = 0.1$, $\Delta = 0.5$, $\mu = 15$, $n = 1$. 
FIGURE 4. (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section \( \Sigma_{\theta=0} \). Parameter set: \( F = 0.1, \Delta = 0.5, \mu = 15, n = 2 \).

FIGURE 5. Extreme output action values [Eq. (12)] for an initial action \( J_0 = 0.5 \). Parameter sets: \( F = 0.01(a), 0.1(b), \mu = 15, n = 1, 2, 3 \).

FIGURE 6. Varying frequency mismatch. Extreme output action values [Eq. (12)] for an initial action \( J_0 = 0.5 \). Parameter sets: \( F = 0.01, \mu = 15, n = 1(a), 2(b), \delta^{(1)} = 0, -0.02, -0.05 \).

FIGURE 7. Efficiency plots [Eq. (15)]. (a) Constant frequency mismatch. Parameter set: \( F = 0.01, \mu n = 15, n = 1, 2, 3 \), (b) Varying frequency mismatch. Parameter set: \( F = 0.01, \mu n = 15, n = 1, 2, 3, \delta^{(1)} = -0.02 \), (c) Effect of a Gaussian action spread. Parameter set: \( F = 0.01, \mu n = 15, n = 1, \delta^{(1)} = 0, J_s = 0.01, 0.03, 0.06 \).
FIGURE 8. (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section \( \Sigma_{\theta=0} \) for two modes being in resonance with different harmonics of the electron cyclotron frequency. Parameter set: \((n_1, n_2) = (1, 2), (F_1, F_2) = (0.01, 0.01), (\mu_1, \mu_2) = (15, 15), (\Delta_1, \Delta_2) = (0.7, 0)\).

FIGURE 9. (a) Numerically and (b) analytically [Eqs. (10)-(11)] obtained Poincaré surface of section \( \Sigma_{\theta=0} \) for two modes being in resonance with the same harmonic of the electron cyclotron frequency. Parameter set: \((n_1, n_2) = (1, 1), (F_1, F_2) = (0.01, 0.01), (\mu_1, \mu_2) = (15, 15), (\Delta_1, \Delta_2) = (0.7, 0)\).

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