ANTIPANE PROBLEM OF CRACK ARREST FOR A CRACKED PIEZOELECTRIC PLATE

R.R.Bhargava and Namita Saxena
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee, 247667

ABSTRACT

A crack arrest model is proposed in this paper for an internally cracked piezoelectric ceramic plate. The plate is poled and unbounded. The internal crack weakening the plate is a hairline straight crack. The infinite boundary of the plate is subjected to anti-plane shear stress and in-plane electrical displacement. The electrical displacement component is applied perpendicular to the rims of the crack. On account of these forces the crack yields both mechanically and electrically. Consequently a slide zone and a saturation zone protrude ahead of each tip of the crack. Under small scale yielding the developed zones are assumed to lie along the length of the crack. To arrest opening of the cracks the developed plastic zones rims are subjected to closing uniform yield point shear stress. And rims of developed saturation zones are subjected to in-plane linearly varying saturation limit electrical displacement. Two cases are investigated when:

Case I: Slide zone length is bigger than the saturation zone length,
Case II: Saturation zone length exceeds the slide zone length.

Each case is solved using Fourier transform technique. Analytic expression for crack sliding displacement, crack opening potential drop and energy release rate have been derived for each case. The case study for these have been presented in graphical form with respect to affecting parameter viz. material constants, crack length, slide zone length and applied loads for different piezoelectric ceramics.

Introduction

Increasing use of piezoelectric ceramic in hi-tech structures has naturally increased the interest in understanding the mode of working and mechanics of failure of such materials. The aspects of cracking in piezoelectric ceramics is widely studies for almost about twenty years now. The stalwarts viz. Sun, Pak, Deeg, Sosa, Suo, Park, Rajapske, McMeeking, Hermann, Shen, Shindo, Narita etc. have given linear piezoelectric treatment for a cracked piezoelectric matrix under different modes and geometries. This is further developed in specific direction of unbounded plate, infinite and semi-infinite strips and rectangular plate by other researchers. An analytic solution for an elliptic cylindrical cavity or a crack inside an infinite piezoelectric medium under combined mechanical- electrical loadings is obtained by Zhang, Qian and Tong [1]. They solved the problem via the Stroh formalism and confirmed their analysis by finite element analysis. Gao and Wang [2] are an easy method for calculating the energy release rate in linear piezoelectric media weakened by a crack. An non-linear electromechanical interfacial fracture for piezoelectric materials is investigated by Shen et al.[3]. The multiscale nature of cracking in ferroelectric ceramic is explored by Sih and Zuo [4] by considering a single dominant crack is where the effect of microcracking could be reflected by a stable crack growth prior to macro crack instability using the strain energy density theory.

Zhou-Cheng and Chen [10] investigated the approach of the crack tip energy release rate for a semi permeable crack full with air/vacuum or silicon oil when electromechanical loads became very large. Zhou and Wu [11] used the non-local theory of elasticity to obtain the behavior of a Griffith crack in functionally graded piezoelectric materials under the anti-plane shear loading conditions.

**Basic Mathematical Formulation**

As is well-known the constitutive equation for piezoelectric plate may be written as

\[
\begin{align*}
\sigma_{xz} &= c_{44} w_{,x} + e_{15} \phi_{,x} \\
\sigma_{yz} &= c_{44} w_{,y} + e_{15} \phi_{,y} \\
D_x &= e_{15} w_{,x} - \varepsilon_{11} \phi_{,x} \\
D_y &= e_{15} w_{,y} - \varepsilon_{11} \phi_{,y}
\end{align*}
\]  

(1)

where

\( \sigma_{xz}, \sigma_{yz} \) = anti-plane stresses

\( D_x, D_y \) = in-plane electrical displacement

\( c_{44} \) = elastic constant of the ceramic

\( e_{15} \) = piezoelectric constant of the ceramic

\( \varepsilon_{11} \) = dielectric constant of the ceramic

\( \phi \) = electric potential

\( , = \) denotes partial differentiation with respect to argument following it

\( w (x, y) \) = out-of-plane displacement.

and

\[ E_i = -\phi_{,i}. \]  

(3)

Substituting from equations (1 and 2) in the equilibrium equations for stresses and electric displacement \( \sigma_{ij} = 0 \) and \( D_{ij} = 0 \), respectively, following simplified governing equations are obtained

\[ c_{44} \nabla^2 w + e_{15} \nabla^2 \phi = 0 \quad \text{and} \quad c_{44} \nabla^2 w - \varepsilon_{11} \nabla^2 \phi = 0 \]  

(4)

Introducing the notation

\[ \bar{t} (x, y) = \left\{ \begin{array}{ll}
\sigma_{yz} & \\
D_y & 
\end{array} \right\}^T \quad \text{and} \quad \bar{u} (x, y) = \left\{ w, \phi \right\}^T. \]  

(5)

and taking Fourier transform of equation (4), using above notations the general solution of equation (4) may be written as

\[ \tilde{u}^+ (s, y) = \begin{cases} 
e^{-sC} & s < 0 \\
e^{-sF} & s > 0 \end{cases} \]  

(6)

where

\( \sim = \) Fourier transform

\( s = \) Fourier transform variable

\( C, F \) = Intermediate 2-component columns

Let jump in displacement and electric potential along \( y = 0 \), is defined as

\[ \Delta \bar{u} (x) = \bar{u}^+ (x, 0) - \bar{u}^- (x, 0) \]  

(7)

where \( +, - \) = superscripts denotes the values of the function as approached from \( y > 0 \) and \( y < 0 \) planes.
For a crack of length 2a, occupying the interval \([-a, a]\) on x-axis in an unbounded piezoelectric ceramic, it is obvious \(\Delta u(x) = 0\) for \(|x| > a\). The dislocation function and dipole density vectors are given by

\[
\bar{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \frac{d}{dx} \Delta u(x)
\]

Taking Fourier transform of equation (7) and using equation (8), one gets

\[
\Delta \tilde{u}(x) = \tilde{u}^+(x, 0) - \tilde{u}^-(x, 0) = \frac{1}{s} \int_{-a}^{a} \tilde{f}(x) e^{ixx} dx
\]

From the continuity condition of tractions at \(y = 0\), one obtains

\[
\tilde{u}^+(x, 0) - \tilde{u}^-(x, 0) = 0.
\]

Taking the Fourier transform of second of equations of equations (1, 2) and applying boundary condition on \(y = 0\), following is obtained

\[
\tilde{t}(s, 0) = s H \tilde{u}(s, 0)
\]

where

\[
H = \begin{pmatrix} c_{44} & e_{15} \\ e_{15} & -e_{11} \end{pmatrix} = \begin{pmatrix} H_{ij} \end{pmatrix}
\]

Taking inverse Fourier transform of equation (11) and using equations (9 and 10), one gets

\[
\tilde{t}(x) = \frac{1}{\pi} H \int_{-a}^{a} \tilde{f}(t) \frac{dt}{t-x} = \begin{pmatrix} t_1^0(x) \\ t_2^0(x) \end{pmatrix}
\]

where

\[ t_1^0(x) = \text{prescribed mechanical boundary condition} \]

\[ t_2^0(x) = \text{prescribed electrical boundary condition} \]

The crack opening displacement, \(\Delta w(x)\), is calculated using

\[
\Delta w(x) = -\int_{-b}^{b} f_1(s) ds
\]

where

\[ f_1(s) = \text{dislocation potential}. \]

Crack opening potential drop, \(\Delta \phi(x)\), is calculated using relation

\[
\Delta \phi(x) = -\int_{-l}^{l} f_2(s) ds
\]

where

\[ f_2(s) = \text{dipole density potential}. \]

Energy release rate, G, computed using formula
\[ G = \lim_{\delta a \to 0} \frac{D}{2\delta a} \int_0^{\delta a} r \Delta\phi (\delta a - r) \, dr + \tau, \Delta w(a) \] 

(15)

The Problem and its Solution

A piezoelectric ceramic exhibiting a hexagonal symmetry occupies \( oxyz \)-planes. The ceramic is poled along oz-direction. The straight hairline crack occupies the interval \([-a, a]\) on ox-axis. The rims of the crack are stress free. The infinite boundary is subjected to the uniform constant anti-shear stress \( \tau_{yz} = \tau_{\infty} \), and in-plane uniform constant displacement \( D_y = D_{\infty} \).

Consequently the crack yields both mechanically and electrically. Hence at each tip of the crack a slide zone and a saturation zone develop. Under small scale yielding each of the zones are assumed to lie along the length of the crack. The slide zones occupy the intervals \([a, b]\) and \([-b, -a]\) on ox-axis. Saturation zones occupy the intervals \([a, l]\) and \([-l, -a]\), respectively. Two cases are considered

Case I: Size of saturation zone exceeds that of the slide zone \( b < l \).

Case II: Length of slide zone is bigger than saturation zone length \( l < b \).

To arrest the opening of the crack, rims of the developed slide zones are subjected to uniform constant yield point anti-plane stress. And the rims of developed saturation zones are subjected to linearly varying in-plane saturation limit electrical displacement \( \frac{f}{a} D_y \).

![Figure 1 Schematic representation of the problem](image)

As is well-known the solution of the traction-free crack problems corresponding to loads at infinity can be obtained through linear superposition of solutions of two problems one for an uncracked body and the other for loading along the crack faces by self-equilibrated tractions.

Case I: When saturation zone exceeds the length of slide zone (b<l)

For the case by superposition, the boundary conditions may be written as

\[
\begin{align*}
t^0_1 = \begin{cases} -\tau_{\infty} & \text{if } |x| < a \\ -\tau_{\infty} + \tau_x & \text{if } a < |x| < b \end{cases} \\
t^0_2 = \begin{cases} -D_{\infty} & \text{if } |x| < a \\ -D_{\infty} + \left(\frac{f}{a}\right)D_x & \text{if } a < |x| < l \end{cases}
\end{align*}
\]

and

\[
f_1(x) = 0 \quad \text{for } |x| > b ; \quad f_2(x) = 0 \quad \text{for } |x| > l.
\]
Using these boundaries conditions and equation (12) following to equations may be written to obtain the solution

\[
\frac{1}{2\pi} H \int_{-b}^{b} \int_{-t-x}^{t-x} dt = G_{ij} f_j^0(x), |x| < b
\]

(16)

\[
\frac{1}{\pi} \int_{-l}^{l} \int_{-t-x}^{t-x} dt = t_2^0(x), |x| < l
\]

(17)

where \( G = H^{-1} = (G_{ij}) \)

The condition of the existence of solution for equations (16 and 17) yield nonlinear equations to determine \( b \) and \( l \) from

\[
\frac{b}{a} = \sec \left( \frac{\pi}{2} \left( \frac{G_{i1} \tau_\infty + G_{12} D_\infty}{G_{12} D_s} \right) \right)
\]

(18)

and

\[
\frac{l}{a} = \left( 4D_s^2 + \pi^2 D_\infty^2 \right)^{\frac{1}{2}} / 2D_s
\]

(19)

Using these conditions finally \( \tilde{f}(x) \) is calculated for this case as

\[
\tilde{f}(x) = \begin{cases} f_1(x) \\ f_2(x) \end{cases}
\]

where

\[
f_1(x) = \frac{G_{21}}{\pi} \left\{ w(x,a,b) - w(-x,a,b) \right\} + \frac{2D_s G_{12}}{\pi a} \cos^{-1} \left( \frac{a}{b} \right) \sqrt{b^2 - x^2} + \frac{D_s G_{12}}{\pi a} \left\{ w(x,a,b) - w(-x,a,b) \right\},
\]

\[
f_2(x) = -\frac{H_{21}}{H_{22}} f_1(x) + \frac{1}{\pi H_{22}} \left[ 2D_s G_{12} \cos^{-1} \left( \frac{a}{l} \right) \sqrt{l^2 - x^2} + \frac{x D_s}{a} \left\{ w(x,a,l) - w(-x,a,l) \right\} \right],
\]

and

\[
w(x,a,b) = \cos h^{-1} \left\{ \frac{b^2 - a^2}{(a-x)b + a} \right\}.
\]

**Case II: Size of slide zone is bigger than saturation zone length \((l<b)\)**

Boundary conditions of the problem are

\[
t_1^0 = \begin{cases} -\tau_\infty & \text{if } |x| < a \\ -\tau_\infty + \tau_s & \text{if } a < |x| < b \end{cases}
\]

\[
t_2^0 = \begin{cases} -D_\infty & \text{if } |x| < a \\ -D_\infty + \left( \frac{t}{a} \right) D_s & \text{if } a < |x| < l \end{cases}
\]

and

\[
f_2(x) = 0 \quad \text{for } |x| > l; \quad f_1(x) = 0 \quad \text{for } |x| > b.
\]
Solving equation (12) under above boundary conditions following two equations are obtained to determine \( f_1(x) \) and \( f_2(x) \)

\[
- \frac{1}{\pi} \int_{-l}^{l} \frac{f_1(t)}{t-x} \, dt = G_{2j} t_j^0(x), \quad |x| < l
\]

\[
- \frac{1}{\pi} \int_{-b}^{b} \frac{H_{1j} f_j(t)}{t-x} \, dt = t_l^0(x), \quad |x| < b
\]

The condition of the existence of solution for equations (20 and 21) yield nonlinear equations to determine \( l \) and \( b \) from

\[
\frac{l}{a} = \sec \left( \frac{\pi}{2} \left( \frac{G_{21} \tau_\infty + G_{22} D_\infty}{G_{21} \tau_s} \right) \right),
\]

\[
\frac{b}{a} = \sec \left( \frac{\pi \tau_\infty}{2 \tau_s} \right).
\]

Dislocation potential \( f_1(x) \) is given by

\[
f_1(x) = \frac{\tau_s}{\pi H_{11}} \left\{ w(x, a, b) - w(-x, a, b) \right\} - \frac{H_{12}}{H_{11}} f_2(x), \quad |x| < b
\]

Equation (20) together with condition (22) determines dipole density vector \( f_2(x) \) as

\[
f_2(x) = \frac{G_{21} \tau_s}{\pi} \left\{ w(x, a, l) - w(-x, a, l) \right\} + \frac{2D_1 G_{12}}{\pi a} \cos^{-1} \left( \frac{a}{l} \right) \sqrt{l^2 - x^2} + \frac{D_2 G_{22}}{\pi a} \left\{ w(x, a, l) - w(-x, a, l) \right\}
\]

**Crack opening displacement and crack opening potential drop**

**Case I:** Crack opening displacement, \( \Delta w(x) \), determined from equation (13) and at the tip \( x = a \) of the crack opening displacement, \( \Delta w(a) \), is given by

\[
\Delta w(a) = \frac{a}{\pi} \left( 2G_{11} \tau_s + D_2 G_{12} \right) w(-a, a, b) + \frac{G_{12} b^2 D_s}{a} \cos^{-1} \left( \frac{a}{b} \right) \begin{bmatrix} \cos^{-1} \left( \frac{a}{b} \right) \cdot \pi \cdot 1 \end{bmatrix} - D_2 G_{12} \sqrt{b^2 - a^2}.
\]

Crack opening potential drop, \( \Delta \phi(a) \), at the crack tip \( x = a \) is calculated using equation (14) and may be written as

\[
\Delta \phi(a) = \frac{1}{a \pi H_{22}} \left[ -\pi a H_{21} \Delta w(a) + l^2 D_s \cos^{-1} \left( \frac{a}{l} \right) \cos^{-1} \left( \frac{a}{l} \right) \cdot \pi \right] + D_s \left\{ a w(-a, a, l) - \pi \sqrt{l^2 - a^2} \right\}
\]

**Case II:** Crack opening potential drop, \( \Delta \phi(a) \), for this case at the tip \( x = a \) is calculated using equation (14) and is given by
\[ \pi a \Delta \phi (a) = (2aG_{21} \tau_s + aD_s G_{22}) w(-a, a, l) + G_{22} D_1 \cos^{-1}\left(\frac{a}{l}\right) \times \left[l^2 \cos^{-1}\left(\frac{a}{l}\right) - \pi\right] - \pi a \sqrt{l^2 - a^2} \] (28)

Crack opening displacement at the tip \( x = a \) is computed using equation (13) may be written as

\[ \pi H_{11} \Delta w(a) = 2a \tau_s w(-a, a, b) - H_{12} \pi \Delta \phi(a). \] (29)

### Energy release rate

**Case I:** Energy release rate calculated using equation (15) and for this case may be written as

\[
G = \frac{(\pi + 2)D_1^2}{16\pi a^2 H_{22}} \left\{ H_{21} G_{12} \left[b^2 + \sqrt{b^2 - a^2}\right] + l^3 \cos^{-1}\left(\frac{a}{l}\right) + a l \sqrt{l^2 - a^2}\right\} + \tau_s \Delta w(a)
\] (30)

where \( \Delta w(a) \) is obtained from equation (26).

**Case II:** For this case energy release rate obtained using equation (15) and may be written as

\[
G = \frac{D_1^2 l G_{22}}{16\pi a^2} \left\{ l^2 \cos\left(\frac{a}{l}\right) - a \sqrt{l^2 - a^2}\right\} (2 - \pi) + \tau_s \Delta w(a)
\] (31)

where \( \Delta w(a) \) is substituted using equation (29).

### Case Study and Discussion

Normalized crack opening displacement and energy release rate for both the cases have been plotted for the ceramics PZT-5, PZT-5H and BaTiO₃.

<table>
<thead>
<tr>
<th>Material</th>
<th>( c_{44} \left(10^{10} \text{ N/m}^2\right) )</th>
<th>( e_{15} \left(\text{C/m}^2\right) )</th>
<th>( \varepsilon_{11} \left(10^{-10} \text{ Cm/V}\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-5</td>
<td>2.11</td>
<td>12.3</td>
<td>81.103</td>
</tr>
<tr>
<td>PZT-5H</td>
<td>3.53</td>
<td>17.0</td>
<td>151</td>
</tr>
<tr>
<td>BaTiO₃</td>
<td>4.3</td>
<td>11.6</td>
<td>112</td>
</tr>
</tbody>
</table>

Table 1: Material constants
Fig. 2 Variation of normalized COD vs. ratio of saturation zone to half crack length

Fig. 2 depicts the variation of crack opening displacement with respect to saturation zone to crack length ratio. It is observed that as this ratio increases the crack opening reduces for all the ceramics. It is further noted that for PZT-5 crack opens less and PZT-5H and BaTiO$_3$ variation follow each other more closely as slide zone to crack length ratio is increased. The crack for both these ceramics open more as compared to that for PZT-5.

Fig. 3 Variation of normalized COD vs. ratio of slide zone to half crack length

Fig. 3 shows the same variation for case II with respect to slide zone. In this case as the ratio increases the crack opens more for all the ceramic. This behavior is opposite to that as seen in case I with respect to saturation zone increases. In this case PZT-5 AND BaTiO$_3$ follow each other closely while PZT-5H moves further away for the increase in slide zone length.

Fig. 4 Variation of normalized energy release rate vs. ratio of saturation zone to half crack length

Fig. 4 shows the variation of energy release rate with increasing saturation zone size to crack length ratio. It is seen that energy dissipated reduces in this case for all the ceramics. Thus the crack will be opening less in mode-III deformations.
Fig. 5 depicts the variation of normalized energy release rate with increase in slide zone length. It is noted as the slide zone increases the energy release rate for this case also increases almost nonlinearly. Therefore crack will open more in this case as compared to that Case I.

Acknowledgments

Authors are grateful to Prof. R. D. Bhargava [(Senior Professor (retd.), Indian Institute of Technology, Mumbai, India)] for his suggestions and encouragement during the course of this work.

References