Combining Thermoelasticity and a Stress Function to Evaluate Individual Stresses around a Near-Edge Hole Located Beneath a Concentrated Load

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ABSTRACT

Individual stresses are determined on and near the edge of a hole which is located below a concentrated edge-load in an approximate half-plane. Experimental thermoelastic data are combined with an Airy’s stress function. Coefficients of the stress function are evaluated from the recorded TSA data and the traction-free conditions on the hole boundary are satisfied by imposing $\sigma_{rr} = \tau_{r\theta} = 0$ on the edge of the hole for all values of the angle $\theta$. This advantageously enables one to reduce the number of coefficients in the stress function series. The method simultaneously smoothes the measured input data, satisfies the traction-free boundary conditions and evaluates individual stresses on, and in the neighborhood of, the edge of the hole.

Introduction

The most serious stresses in a component frequently occur at geometric discontinuities. Theoretical solutions are seldom available for finite geometries, and purely theoretical or numerical methods typically necessitate accurately knowing the far-field geometry and boundary conditions. The latter information can be unavailable. Experimental techniques therefore continue to be important for determining the stresses associated with geometric discontinuities in finite plane-stressed members. Although it can be advantageous to combine measured information with other supplementary experimental or numerical techniques, it is most convenient to acquire all the necessary measured data utilizing a single experimental method.

The present approach employs an Airy’s stress function and thermoelasticity [1] to determine the stresses in a finite plate containing a near-edge circular hole, the plate being subjected to an edge-load above the hole, Figure 1. Interest in the problem is motivated by the manufacture of circuit boards and buried structures. For practical consideration, the semi-infinite plate (half-plane) is approximated here by a fairly large, finite plate, Figure 1, supported along the bottom edge, CDC’. From thermoelastically-measured isopachic data and incorporating limit boundary conditions, one is able to evaluate the coefficients of stress expressions which are derived from a general relevant Airy’s stress function. Consequently, the individual stress components can be obtained at least in the region of the prime interest, i.e., on, and adjacent to, the edge of the hole.
The Airy's stress function, $\phi$, is the solution of the bi-harmonic equation $\nabla^4 \phi = 0$ \cite{2}. While $\phi$ might consist of numerous terms, many of these terms can often be eliminated by various conditions and/or arguments (e.g., symmetry; single-valued stresses, strains, displacements; equilibrium at boundaries; boundedness at the origin or infinity). Assuming geometric and loading symmetry about the x-axis of Figure 1, a relevant general form of stress function $\phi_{cen}$ here is obtained by combining stress functions for a finite plate which can accommodate a hole and a semi-infinite plate subjected to a concentrated edge load per unit thickness \cite{3}, $P$, i.e.,

$$\phi_{cen} = \frac{P}{\pi} \cdot \tan^{-1}\left(\frac{r \cdot \sin \theta}{D - r \cdot \cos \theta}\right) \cdot r \cdot \sin \theta + a_n \cdot \ln r + c_n \cdot r^2 + \left(a_n \cdot r + c_n \cdot r^2 + \frac{d_n}{r} \cdot r^2 \right) \cdot \cos \theta$$

$$+ \sum_{n=2,3,4,\ldots} \left(a_n \cdot r^n + b_n \cdot r^{n+2} + c_n \cdot r^n + d_n \cdot r^{n-2}\right) \cdot \cos(n \cdot \theta)$$

(1)

Such that $P$ is the specific load equal to the applied concentrated edge force $P^*$ divided by the thickness, $t$, of the plate, $D$ is the location of the center of the hole from the top of the plate, $R$ is the radius of the hole and $\theta$ is measured clockwise from the longitudinal x-axis, Figure 1. Differentiating Eq. (1) according to,

$$\sigma_r = \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \phi}{\partial \theta^2} \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial \theta^2} \quad \tau_{r\theta} = -\frac{\partial}{\partial r} \cdot \left(\frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta}\right)$$

(2)

gives the individual stresses. For isotropy,

$$S^* = KS \quad \text{and} \quad S = \sigma_r + \sigma_\theta$$

(3)
where \( S^* \) is the thermoelastic signal, \( K \) is the thermo-mechanical coefficient and \( S = \sigma_{rr} + \sigma_{\theta\theta} \) is the isopachic stress. Individual stresses can therefore be evaluated if the Airy’s coefficients of Eq. (1) and \( P \) are known, even without knowledge of constitutive information (but assuming elastic isotropy) or far-field geometry or boundary conditions. One could also treat load \( P \) (per thickness) as an unknown.

Advantages of the described method here include knowledge of the loading, material properties, or boundary conditions beyond the traction-free conditions at the hole is not required. Not knowing the far-field geometry or external boundary conditions, including the concentrated load, \( P \) (per unit thickness), makes this an inverse problem. Emphasizing the situation around the hole (typically the region of practical interest) such that one imposes the traction-free conditions on the edge of the hole, along with measured isopachic data near the hole, enables solution of the problem at (and in the neighborhood of) the circular geometric discontinuity, irrespective of the external shape or form or magnitude of loading along the external edges of the plate, ABCDC'B'A', Figure 1. Since measured stress (strain or displacement) information is often unreliable near a boundary, the technique provides reliable stresses on the edge of the hole without using recorded data on, or immediately near, the edge of the hole. Figure 2 is a thermoelastic (TSA) image of the measured temperature information.

![Thermoelastic Image](image)

**Figure 2.** Thermoelastic image of the loaded plate of Figure 1

**Individual Stresses**

The independent coefficients of Eq. (1) can be reduced by *analytically/continuously* imposing traction-free conditions, \( \sigma_{rr} = \tau_{r\theta} = 0 \) at the edge of the hole, Figure 1. Incorporating these boundary conditions at the hole in the stress expressions enables one to acquire accurate stresses utilizing fewer coefficients (stresses now only depend on coefficients \( c_0, d_1, c_2, d_2, c_3, d_3, \) and \( c_n \) and \( d_n \) for \( n>3 \), i.e., only about half as many coefficients than if these conditions were satisfied *non-analytically* at the hole) and potentially to reduce the amount of measured data needed. While having sufficient data is seldom a concern with TSA, it can be with other experimental methods. Imposing traction-free conditions at the edge of the hole also results in identical coefficients existing in the stress term, \( S \), of Eq. (3). All of the individual stresses can therefore be obtained from TSA, without additional experimental data.

**Evaluating the Airy’s Coefficients**

From discretely known experimental stress information \( (S = S^*/K = \sigma_{rr} + \sigma_{\theta\theta}) \) measured by TSA, and if the traction-free boundary conditions, \( \sigma_{rr} = \sigma_{\theta\theta} = 0 \), are *analytically* incorporated on the boundary of the hole, a set of linear isopachic equations containing unknown Airy’s coefficients of Eqs. (1) and (2) can be formed,
Determining a Suitable Number of Airy’s Coefficients

Figure 3 indicates the source locations of 300 TSA input values from Figure 2, used here for analyzing the stresses around the hole in the plate of Figure 1. The following results based on the present TSA-determined Airy’s coefficients are denoted as TSA(Cen). Although all utilized TSA data originated at least four pixels away from the edge of the hole and the top edge of the plate of Figure 1, the approach is able to evaluate stresses at the hole boundary without employing the measured temperature information near the edge. Having the 300 measured isopachs \( S = S/K \), one is able to formulate the least-squares matrix equation \( A \mathbf{c} = \mathbf{d} \) of Eq. (4), where \( A \) contains 300 isopachic expressions of the form of Eq. (3) with \( k \) unknown Airy’s coefficients for the 300 TSA data locations shown in Figure 3. Vector \( \mathbf{c} \) contains \( k \) Airy’s coefficients and vector \( \mathbf{d} \) is composed of the 300 TSA values of \( S \) at the respective locations of Figure 3 associated with the 300 isopachic equations in matrix \( A \).

![Figure 3. 300 source locations of TSA measured input data](image)

Supported by other information, the number of coefficients to retain was determined by the evaluation for the Airy’s matrix condition number and \( RMS \) values of TSA measured \( S \) and calculated isopachs for each of the set of 13 Airy’s matrices \( A \) containing 300 isopachic equations, Eq. (5), based on the thermoelastically recorded \( S \) at the 300 locations of Figure 3 with the various numbers of coefficients from 1 to 25, Figure 4 [4 - 6].

If the Airy’s matrix equation \( A \mathbf{c} = \mathbf{d} \) of Eq. (4) is overdetermined, i.e., \( m > k \), where \( m \) is the number of equations and \( k \) is the number of coefficients, the linear matrix equation will be solved by the least-squares process. Thereby, multiplying the original matrix \( A \) by the least-squares evaluated \( \mathbf{c} \), gives the calculated isopachs \( \mathbf{d}' \), which are typically not exactly the same as \( \mathbf{d} \). It is desirable that the values of the \( RMS \) between the evaluated isopachic data \( \mathbf{d}' \) and thermoelastically measured \( \mathbf{d} \) be small.
Figure 4 plots the RMS values versus different numbers of coefficients \( k \), based on the 300 thermoelastic data at the locations of Figure 3 (i.e., \( m = 300 \) in Eq. (5)). This Figure indicates \( k = 15 \) to be a satisfactory choice. Using more coefficients does not reduce the RMS (Root Mean Square) value much further. Moreover, \( k > 15 \) would entail more calculations and a bigger condition number for the analytical matrix, which can increase the chances of the matrix becoming singular. It also implies a possible need for more measured input data. Thus, based on the 300 TSA data points of Figure 2 and \( m = 300 \), \( k = 15 \), and \( N = 7 \) (i.e., \( 3 + (k-7)/2 = 7 \)), solving the overdetermined Airy’s matrix equation \( Ac = d \) of Eq. (4) by a least-squares process gives the values in the coefficient vector, \( c = (c_0, d_1, c_2, d_2, b_3, c_3, d_3, c_4, d_4, \ldots, c_7, d_7) \). Having evaluated the Airy’s coefficients, i.e., the 15 TSA(Cen)-determined Airy’s coefficients, the individual stresses become available, including those on the edge of the hole.

![Figure 4. RMS values for different numbers of coefficients employed in matrix equation \( Ac = d \) of Eq. (4)](image)

Figure 5 compares TSA measured \( S^* \) with those predicted by the described approach, TSA(Cen), along horizontal line \( ab \) and vertical line \( cd \) of Figure 6. As well as substantiating the selection of \( k = 15 \), Figure 5 demonstrates the unreliability of the recorded TSA data within the first three or four pixels from the boundary of the hole.

![Figure 5. TSA measured \( S^* \) compared with TSA(Cen)](image)
FEM Analysis

In addition to using FEM (ANSYS)-predicted stresses with which to compare the TSA (TSA-determined Airy coefficients) determined stresses (i.e., TSA(Cen)), FEM-predicted values of S (simulated TSA isopachs) were also employed. Results based on ANSYS-generated isopachic values of $\sigma_{rr} + \sigma_{\theta\theta}$, to simulate TSA inputs are denoted as TSA(ANS_Cen). The FE model was meshed by 2621 four-sided, eight-node, isoparametric elements (ANSYS solid 82 element) with 8118 nodes in total. Small elements were utilized in the high stress regions. Elements between the load point and the top edge of the hole are as small as 0.32 mm (0.0125") and the edge of the hole is meshed by 80 elements (element size is approximately 0.374 mm (0.0147"). Geometric and loading symmetry about the x-axis enables FE modeling of only one half of the plate of Figure 1. Unlike TSA (the described approach), FEM requires complete boundary information. A roller constraint was therefore applied along the lower edge of the plate, CDC', of Figure 1.

The TSA(ANS_Cen) calculations use FEM-simulated input values of $S$ at 312 source locations along three concentric arcs having $r$ approximately equal to 12.19 mm (0.48"), 14.48 mm (0.57"), and 16 mm (0.63"), respectively. Recognizing that measured TSA values are unreliable on and immediately near edges, the selected ANSYS nodes are at least 2.54 mm (0.1") away from the boundary of the hole and the top edge of the plate. Using these 312 FE generated isopachic magnitudes of $\sigma_{rr} + \sigma_{\theta\theta}$, the number of coefficients was determined by employing essentially the same approach utilized previously for the 300 input TSA-determined values of $S$. However, for TSA(ANS_Cen), $A$ is an $m (=312)$ by $k (=15)$ matrix containing a set of
312 linear isopachic equations with 15 independent variables at each of the 312 source locations and vector \( d \) is composed of 312 ANSYS-generated isopachic values corresponding to the isopachic equations in matrix \( A \).

**Thermoelastic Stress Analysis (TSA)**

The 6061 T6511 aluminum plate of Figure 1 was sprayed with Krylon flat black paint prior to TSA testing to enhance radiation uniformity and emissivity of the material. The plate was subjected to a sinusoidal force varying between 222.4 N (50 lb) and 1112 N (250 lb) at a rate of 20 Hz using a 88.96 kN (20,000 lb) capacity MTS loading system. TSA data was recorded (2 minute duration) by a nitrogen-cooled Stress Photonics DeltaTherm DT 1410 infrared camera with a sensor array of 256 horizontal x 256 vertical pixels, see Figure 2. The pixel size is approximately 0.7mm (0.03″), which resulted in a total of approximately 14,000 data values being recorded for the entire plate. The value of the thermoelastic coefficient, \( K = 320 \text{ U/MPa} \) (2.21 U/psi), of Eq. (3) was determined from a separate uniaxial tensile coupon.

**Results**

Figure 7 shows the normalized hoop stresses at the edge of the hole. Figures 9 through 11 contain individual stresses further away from the hole boundary, at the locations shown in Figure 8, where \( r \) ranges from \( r = 1.13 R \) to \( 1.21 R \). All actual stresses are normalized by the uniform stress \( \sigma_0 = 1.05 \text{ MPa} = 152.4 \text{ psi} \) \((P^*=\text{ pounds})/\text{gross cross-sectional area of 3.5″ by 3/8″ thick), Figure 1. There is good agreement between the TSA(Cen), ANSYS and TSA(ANS_Cen) results. The slight discrepancy between TSA(Cen) and ANSYS-based predictions is probably caused by the apparent small in-plane bending involved in the experimentally loaded TSA specimen, whose effects are excluded in the FEM. Whereas the FEM assumes \( P^* \) is a point-contact load, the physical loading is not exactly along a line. However, this appears to have virtually no consequence on the stresses near the hole. Due to space limitations, many details and equations have been omitted from this manuscript. They will be included in Ms. Lin’s final thesis document.

![Figure 7. Normalized hoop stress around the boundary of the hole.](image)

![Figure 8. Selected data locations for stress component determination](image)

![Figure 9. Normalized \( \sigma_r \) for selected locations shown in Figure 8](image)
Summary, Discussion, and Conclusion

The present study emphasizes the ability to obtain individual stress components using a single experimental method, TSA, combined with a stress function and without knowing the complete/external boundary conditions, material properties or loading information. Although FEA can evaluate stress values without experimental aid, it necessitates accurate information on the complete boundary conditions – details which are often unavailable. Moreover, applied thermoelastic stress analysis requires less elaborate specimen preparation than many of the alternative experimental methods, permits the structure to function in its normal environment, and records (and processes) vast amounts of data quickly. While the problem addressed here involves a square, finite loaded plate containing a near surface hole, the described approach is applicable to a wide array of problems.

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References