Determining Individual Stresses around a Near-Edge Hole in a Plate Subjected to an Offset Load using Thermoelasticity

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ABSTRACT

The paper combines an Airy’s stress function in real, polar coordinates with the experimental method of thermoelastic stress analysis (TSA) to determine the individual stresses in an aluminum half-plane which contains a near-edge circular hole, the plate being subjected to a concentrated edge load away from the hole. The coefficients of the stress function are evaluated from the thermoelastically measured data using the least squares method. Imposing the traction-free conditions analytically, rather than discretely, on the edge of the hole significantly reduces the number of coefficients one must retain in the stress function, the number of equations involved in the least squares process, and in some cases the amount of measured input data needed. Problems such as the present one can also be solved from photoelastically recorded isochromatics. However, whereas the latter approach necessitates time-consuming iterative non-linear least squares, the present TSA-based scheme only requires linear least squares. TSA, which can be applied to the actual material of interest (no model or coating is needed, other than perhaps being painted flat black to enhance the uniformity and emissivity of the material) benefits from the availability of contemporary commercial systems capable of providing extensive amounts of data in a matter of minutes.

Introduction

The most serious stresses in a component frequently occur at geometry discontinuities. Theoretical solutions are seldom available for finite geometries, and purely theoretical or numerical methods typically necessitate accurately knowing the far-field geometry and complete boundary conditions. The latter information is often unavailable. Experimental techniques are therefore important for determining the stresses associated with geometric discontinuities in plane-stressed finite components. Although the in-plane stress components can be obtained by combining measured information with other supplementary experiments, it is convenient to acquire all the necessary measured data from a single experimental method.

The present approach employs an Airy’s stress function and thermoelasticity \cite{1} to determine the stresses in a finite plate containing a near-edge circular hole and subjected to an offset edge load, Figure\textsuperscript{1}. The motivation for this problem comes from circuit boards and buried structures. For practical considerations, the semi-infinite plate (half-plane) is approximated here by a fairly large, finite plate, Figure \textsuperscript{1}, supported along the bottom edge, CDC’. From thermoelastically-measured isopachic data and incorporating limit boundary conditions, one is able to evaluate the coefficients of the stress expressions which have been derived from a general relevant Airy’s stress function. Consequently, the individual stresses can be obtained at least in the region of prime interest, \textit{i.e.}, on, and adjacent to, the edge of the hole. Problems such as the present one can also be solved by Wang’s methods \cite{2-5} and photoelastically recorded isochromatics. However, whereas the latter approach necessitates iterative non-linear least squares, the present TSA-based scheme only requires linear least squares and is applicable to irregularly-shaped finite geometries.
Expression of Stress Function

The Airy's stress function, $\phi$, is the solution of the bi-harmonic equation $\nabla^4 \phi = 0$ (i.e., compatibility) [6]. A relevant general form for $\phi_{\text{offset}}$ here is obtained by combining stress functions for a finite plate which can accommodate a hole and a semi-infinite plate subjected to a concentrated edge load [7], $P$, i.e.,

\[
\phi_{\text{offset}} = -\frac{P}{\pi} \tan^{-1}\left(\frac{r \cdot \sin \theta - E}{D - r \cdot \cos \theta}\right) \cdot (r \cdot \sin \theta - E) + a_0 + b_0 \cdot \ln r + c_0 \cdot r^2 + A_0 \cdot \theta
\]

\[
+ \left(a_1 \cdot r + \frac{c_1}{r} + d_1 \cdot r^3\right) \cdot \sin \theta + \left(a_1' \cdot r + \frac{c_1'}{r} + d_1' \cdot r^3\right) \cdot \cos \theta
\]

\[
+ \sum_{n=2,3,4} \left[a_n \cdot r^n + b_n \cdot r^{(n+2)} + c_n \cdot r^{-n} + d_n \cdot r^{-(n-2)}\right] \cdot \sin(n \cdot \theta) + \sum_{n=2,3,4} \left[a_n' \cdot r^n + b_n' \cdot r^{(n+2)} + c_n' \cdot r^{-n} + d_n' \cdot r^{-(n-2)}\right] \cdot \cos(n \cdot \theta)
\]

(1)

where $P$ is the specific load equal to the concentrated edge force $P^*$ divided by the thickness $t$ of the plate, $D$ is the location of the center of the hole below the top of the plate, $R$ is the radius of the hole, $E$ is the horizontal distance between the center of the hole and load $P^*$, and $\theta$ is measured clockwise from the vertical $x$-axis, Figure 1. Differentiating Eq. (1) according to

\[
\sigma_\theta = \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \quad \sigma_\rho = \frac{\partial^2 \phi}{\partial r \partial \theta} \quad \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta}\right)
\]

(2)

gives the individual stresses. For isotropy,

\[
S^* = K^* S \quad \text{and} \quad S = \sigma_\rho + \sigma_\theta
\]

(3)
where $S^*$ is the thermoelastically-detected signal, $K$ is the thermo-mechanical coefficient and $S = \sigma_{\tau} + \sigma_{\theta\theta}$ is the isopachic stress or the first stress invariant. Individual stresses can therefore be evaluated if the Airy's coefficients and $P$ are known, and without knowledge of constitutive information (but assuming elastic isotropy) or external geometry or boundary conditions. One could also treat load per unit thickness, $P$, as a variable.

Some of the advantages of the described method are that knowledge of the loading, material constitutive properties, or boundary conditions beyond the traction-free conditions at the hole are not required. The lack of the need to know the distant geometry or external boundary conditions, including the concentrated load per unit thickness, $P$, renders this an inverse problem. Emphasizing the situation around the hole (typically the region of practical interest) such that one imposes the traction-free conditions on the edge of the hole in combination with measured isopachic data near the hole, enables one to solve the problem at (and in the neighborhood of) the hole, irrespective of the external shape or form or magnitude of loading along external boundary ABCD′C′B′A′ of Figs. 1 and 2. Since measured stress (strain or displacement) information data is often unreliable on, or immediately near, edges, the technique provides reliable stresses on the boundary of the hole without using unreliable measured data at these locations.

![Thermoelastic image of the loaded plate of Figure 1](image)

**Figure 2.** Thermoelastic image of the loaded plate of Figure 1

**Individual Stresses**

The independent coefficients of Eq. (1) can be reduced by analytically/continuously imposing traction-free conditions, $\sigma_{\tau} = \tau_{\theta\theta} = 0$ at the edge of the hole. Incorporating these boundary conditions at the hole in the stress expressions provides accurate stresses utilizing fewer coefficients (stresses now only depend on variables $c_0$, $d_1$, $d'_1$, $d'_2$, $d_3$, $d'_3$, $d_5$, $d'_5$, $A_0$, $c_n'$, $d_n'$ for $n>3$), and $b_n$, $d_n$ for ($n>1$) (i.e., only about half as many coefficients for a non-analytical study), and potentially reducing the amount of measured data needed. While having sufficient measured data is seldom a concern for TSA, it can be with other experimental methods, for instance strain gages. Imposing traction-free conditions at the edge of the hole also results in identical coefficients existing in the thermoelastic stress expression, $S$, of Eq. (3), so all of the individual stresses can be obtained from TSA without involving far-field conditions or additional experimental data.

**Evaluating the Airy’s Coefficients**

From discretely known experimental stress information ($S = S^*/K = \sigma_{\tau} + \sigma_{\theta\theta}$) measured by TSA, and if the traction-free boundary conditions, $\sigma_{\tau} = \sigma_{\theta\theta} = 0$, are analytically incorporated on the boundary of the hole, a set of linear isopachic equations containing unknown Airy's coefficients of Eqs. (1) through (3) can be formed,
\[ A = \begin{bmatrix} S_{r_0} & c_0 \\ S_{r_1} & d_1 \\ \vdots & \vdots \\ S_{r_m} & d_m \end{bmatrix} \]

\[ \begin{bmatrix} c_0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_m \end{bmatrix} \]

\[ \text{Eq. (4)} \]

\( \mathbf{A} \) is an \( m \) by \( k \) matrix composed of a set of \( m \) isopachic equations with \( k \) Airy's coefficients, \( \mathbf{c} \) is a vector consisting of the \( k \) unknown Airy's coefficients, and vector \( \mathbf{d} \) contains thermoelastically-measured values of \( S \) corresponding to the set of equations in the isopachic matrix \( \mathbf{A} \). Knowing \( \mathbf{A} \) and \( \mathbf{d} \), Eq. (4) can be solved to evaluate the Airy's coefficients, \( \mathbf{c} \). Once the coefficients are obtained, the individual stresses are available. It will be shown subsequently that based on 849 TSA-measured input values of \( S \), Figure 3, \( k = 25 \) is suitable for the present TSA analysis.

**Determining a Suitable Number of Airy's Coefficients**

Figure 3 indicates the source locations of 849 TSA input values from Figure 2 used here for stress analyzing around the hole of the plate of Figure 1. The latter results, based on the described approach of TSA-determined Airy's coefficients, are referred to as TSA(Offset). Although all utilized TSA data originate at least four pixels away from the edge of the hole and the top edge of the plate of Figure 1, the technique is able to evaluate stresses at the edge of the hole without using the unreliable measured edge stress information. For the TSA image of Figure 2, the space between pixel centers is 0.7 mm (0.03”). Having the 849 measured isopachs \( S (= S^*/K) \), one is able to formulate the matrix equation \( \mathbf{A} \mathbf{c} = \mathbf{d} \) of Eq. (4), where \( \mathbf{A} \) contains 849 isopachic expressions of the form provided by Eq. (3) with \( k \) unknown Airy’s coefficients for the 849 TSA recorded data of Figure 3. Vector \( \mathbf{c} \) contains \( k \) Airy’s coefficients and vector \( \mathbf{d} \) is composed of the 849 TSA values of \( S \) at the respective locations of Figure 3 associated with the 849 isopachic equations in matrix \( \mathbf{A} \).

![Figure 3. 849 source locations of TSA measured input data](image)

Supported by other information, the number of coefficients to retain was determined by evaluating the Airy's matrix condition number and RMS (Root Mean Square) values of TSA measured \( S \) and calculated isopachs for each of the set of 11 Airy’s matrices \( \mathbf{A} \) containing 849 isopachic equations, Eq. (5), based on the thermoelastically recorded \( S \) at the 849 locations of Figure 3 with the numbers of coefficients varying from 1 to 41, Figure 4 [8-10].
If the Airy’s matrix equation $Ac = d$ of Eq. (4) is over determined, i.e., $m > k$, where $m$ is the number of equations and $k$ is the number of coefficients, the linear matrix equation will be solved in the least-squares process to result in smooth solutions. However, multiplying the original matrix $A$ by the least-squares obtained $c$ gives the calculated isopachs $d'$ which are typically not exactly the same as $d$. It is desired that the value of the RMS between the evaluated isopachic data $d'$ and the thermoelastically measured $d$ be small.

$$
(d' - d)_{RMS} = \sqrt{\frac{\sum_{i=1}^{m} (d_i' - d_i)^2}{m}}
$$

Figure 4 plots the RMS values versus different numbers of coefficients $k$, based on the 849 thermoelastic data of Figure 3 (i.e., $m = 849$ in Eq. (5)). This graph, Figure 4, indicates $k = 25$ to be a satisfactory choice. Using more coefficients does not reduce the RMS value much further. Moreover, $k > 25$ would entail more calculations, increase the matrix $A$ condition number and imply a possible need for more measured input data. Thus, based on the 849 TSA measured data of Figs. 3 and 4, and $m = 849$, $k = 25$, and $N = 6$ (i.e., $(k-1)/4 = 6$), solving the over-determined Airy's matrix equation $Ac = d$ of Eq. (4) by the least-squares process gives the values of the coefficient vector $c = (c_0, d_1, c_2', d_2', b_3', c_3', d_3', A_0, c_4', d_4', ..., c_6', d_6', b_2, d_2, b_6, d_6)$. Having evaluated the Airy’s coefficients existing in the stress expressions, i.e., the 25 TSA(Offset)-determined coefficients, the magnitudes of the individual stresses are available, including on the edge of the hole, even without employing measured stress information near and on the boundary of the hole.
Figure 5. Isopachs $S^*$ measured by TSA and predicted by the described approach TSA(Offset) with $k = 25$ along (A) lines $ab$ and $a'b'$ and (B) lines $cd$ and $c'd'$ of Figure 6

Figure 6. Source locations of $S^*$ for TSA measurement and the reconstructed values, TSA(Offset), of Figure 5

**FEM Analysis**

In addition to using FEM (ANSYS)-predicted stresses with which to compare the TSA (TSA-determined Airy’s coefficients) determined stresses (i.e., TSA(Offset)), FEM-predicted input values of $S$ (simulated TSA isopachs) were employed. Results based on ANSYS-generated isopachic values of $\sigma_{rr} + \sigma_{\theta\theta}$ to simulate TSA inputs are denoted as TSA(ANS_Offset). The FE model of Figure 1 was meshed by 3896 four-sided, eight-node isoparametric elements (ANSYS solid 82 element) with 13332 nodes in total. Small elements were utilized in the high stress regions. Elements between the top edge of the plate and circular hole are as small as 0.64 mm (0.025") and the edge of the hole is meshed by 99 elements (element size is approximately 0.61 mm (0.024")). The lack of geometric or loading symmetry requires a full FE model of Figure 1. Unlike TSA (the described approach), FEM requires complete boundary information. A roller constraint (no vertical motion) was therefore applied along the lower edge of the plate, CDC', of Figure 1.

The TSA(ANS_Offset) calculations use FEM-simulated input values of $S$ at 288 source locations along three perimeters having $r$ approximately equal to 12.45 mm (0.49"), 14.22 mm (0.56"), and 16 mm (0.63"), respectively. Recognizing that measured TSA values are unreliable on and immediately near edges, the selected ANSYS nodes are at least 2.54 mm (0.1") away from the boundary of the hole and the top edge of the plate. Using these 288 FE generated isopachic magnitudes of $\sigma_{rr} + \sigma_{\theta\theta}$, one is able to solve the Airy’s matrix equation $Ac = d$ of Eq. (4), where $A$ is an $m$ (= 288) by $k$ (= 25) matrix containing a set of 288 linear isopachic equations with 25 independent coefficients at each of the 288 FE source locations and vector $d$ is composed of the 288 ANSYS-generated isopachic values corresponding to the isopachic equations in matrix $A$. After the coefficient vector, $c$, containing all of the coefficients found in the stress expressions is obtained, the components of stress based on TSA(ANS_Offset) are subsequently available.
Thermoelastic Stress Analysis (TSA)

The 6061 T6511 aluminum plate of Figure 1 was sprayed with Krylon-flat black paint prior to TSA testing to enhance radiation uniformity and emissivity. The plate was subjected to a sinusoidal force varying between 222.4 N (50 lb) and 1112 N (250 lb) at a rate of 20 Hz using an 88.96 kN (20,000 lb) capacity MTS loading system. TSA data were recorded (2-minute duration) by a nitrogen-cooled Stress Photonics DeltaTherm DT 1410 infrared camera with a sensor array of 256 horizontal x 256 vertical pixels, Figure 2. The pixel spacing was approximately 0.7mm (0.03″), resulting in slightly more than 15,000 data values being recorded from the entire plate. Calibration of the thermoelastic coefficient, \( K = 320 \text{U/MPa} \) (2.21, U/psi) of Eq. (3) was determined from a separate uniaxial tensile coupon.

Results

Figure 7 shows the normalized hoop stresses around the edge of the hole. Figures 9 through 11 contain individual stresses further away from the hole boundary at the locations shown in Figure 8, where \( r = 1.2 R \) to \( 1.27R \). All actual stresses are normalized by the uniform stress \( \sigma_0 = 1.05 \text{ MPa} = 152.4 \text{ psi} \) (\( P^* \)= pounds)/gross cross-sectional area of 3.5" wide by 3/8" thick), Figure 1. Excellent agreement between TSA(Offset), and the FEM-based results of TSA(ANS_Offset) and ANSYS, Figs. 7 and 9 through 11, shows the validity of the present approach. Whereas FEM assumes \( P^* \) is a contact point load, the physical loading is not confined to a line. As often occurs, FEM does not model the actual physical loading exactly here, but this probably has virtually no effect at the hole. However, the described TSA scheme overcomes this difficulty. Besides, by analytically satisfying traction-free conditions at the hole boundary, the experimental approach is independent of the external boundary conditions.

Due to space limitations, many details and equations are omitted from this manuscript. They will be included in Ms. Lin’s thesis document.
Summary, Discussion, and Conclusion

The present study provides individual stress components using a single experimental method, combined with a stress function and without knowledge of external geometry, material properties, or distant loading information. Although FEM is able to evaluate stresses without experimental measurements, it necessitates accurate information of the complete geometry and boundary conditions, information which is often unavailable. Moreover, unlike some tedious experimental methods (for instances, strain gages, moiré), that TSA requires minimal specimen preparation, can analyze a member operating in its normal environment, and can do so accurately and expeditiously, indicates the applicability of the present TSA approach to other practical problems.

This paper analytically imposes traction-free conditions at the edge of the hole in order to efficiently reduce the independent variables existing in the stress equations derived from the relevant Airy’s stress function and the measured data needed for solving the matrix equation $Ac = d$. The present approach is able to stress analyze quite general and complicated geometries. It might be convenient in some cases to impose boundary conditions (perhaps at a cutout) discretely/point-wise, rather than analytically. This will change the make-up of matrix $A$ by adding normal and shear stress equations. However, the components of stress can still be obtained without knowing the constitutive properties, or distant geometry or loading. The lack of loading symmetry here implies many more Airy’s coefficients in the stress function (and hence in the expressions for the stresses) than if there were symmetry. The current technique significantly reduces the number of these coefficients.

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