Detection of multiple cracks in cantilever beams using frequency measurements

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ABSTRACT
This paper examines the use of natural frequency measurements for detection and assessment of multiple cracks in beams using the energy approach. The damage is modelled as a spring whose stiffness is inversely proportional to the damage size. The previous applications of the energy approach in this regard have relied on analytical modelling of the structure to determine the vibration mode shapes. This has the disadvantage of being applicable only to simple structures with simple boundary conditions. It is proposed in the present work that discrete values of structure deflections, such as those measured by experimental modal analysis or obtained by finite element modelling, can be employed to determine the location and assess the size of the damage. The applicability of this approach is demonstrated through a numerical study, employing mode shapes obtained by finite element analysis to determine locations and sizes of multiple through thickness centre cracks in cantilever beams with greater than 95 percent accuracy. This paves the way for implementation of the technique to more complex structures without the need to develop or employ complex analytical models.

Introduction
With both Civilian and military operators seeking more cost effective and efficient means of aircraft maintenance, the new approach to airframe structural maintenance involves implementation of on-line health monitoring systems, where the structure is monitored for damage continuously in real time. This reduces the aircraft’s down time assuring an uninterrupted operation and further reduces the need for skilled people involved in conventional non-destructive evaluation. Though techniques like use of fiber optic sensors and acoustic emission are being explored, the most promising technique for on-line health monitoring is the measurement of vibration parameters, because they are highly sensitive to the initiation and progression of damage in the structure. It is quite well known that a crack or any damage in a structure changes its dynamic characteristics, viz. natural frequencies, mode shapes and damping. The changes in these dynamic properties depend on the location and size of damage. Hence, by monitoring the change in any or all of these parameters, damage can be characterised. Frequency can be measured with least uncertainty and high repeatability, using just one sensor [1], compared to mode shapes, which is highly dependent on the number of sensors used and distribution of sensors.

One of the earliest applications of the vibration method was reported by [2], where damage was detected in an off-shore light structure. In applying frequency measurements to crack detection, changes in measured frequencies due to the damage in the structure have to be compared to predicted frequency changes, which may be obtained either by numerical simulation or mathematical modelling. In the latter method, cracks are modelled as torsional springs whose stiffness k is inversely proportional to the extent of damage, and account for the discontinuity in slope at the location of the damage. This concept has been studied by [3, 4] employing solution of the governing equation of equilibrium to determine the modal shape function for the beam, and by [5, 6, 7], who use the energy approach which has the advantage of being applicable to detection of multiple cracks.

In the energy approach, the key parameter representing the ratio of the energy of the spring to the energy of the beam, g(β), is calculated from the mode shapes of the undamaged beam, which are obtained from Euler-Bernoulli beam vibration theory with the appropriate boundary conditions, either using symbolic math [5, 6] or manual solution [7]. In either case analytical modelling of the beam and its boundary conditions is necessary to obtain the mode shapes, which is not easily applicable to real complex structures.
In the present paper, it is proposed that mode shapes measured by experimental modal analysis of the undamaged structure can be employed to determine the strain energy parameter. This has the advantage that mathematical modelling of the structure is not necessary, making it viable for application to real complex structures. At the same time, since the method does not rely on measured differences in mode shapes or its derivatives as in traditional damage detection techniques using mode shapes, but only on the mode shape of the undamaged structure, it is not susceptible to uncertainties due to noise or changes in boundary or environmental conditions between measurements taken on the undamaged and damaged structures. The method is in essence a hybrid of the frequency measurement and mode shape techniques, since it uses measured frequency changes and measured mode shapes of the undamaged beam, but does not require numerical or mathematical modelling of the structure and its boundary conditions. The applicability of this hybrid approach is verified by identifying the locations and assessing the lengths of the two centre cracks in a cantilever beam. For validation purposes, the frequencies of the damaged and undamaged beams and the mode shapes of the latter are obtained by numerical simulation using commercial Finite Element Analysis (FEA) software, ANSYS 10.

THEORETICAL FORMULATION USING SPRING MODEL FOR CRACK

The crack is modelled as a discontinuity in stiffness represented by a massless, rotational spring with stiffness, \( k_r \), which is inversely proportional to the size of crack [4]. In the present case, the crack is assumed to be full depth (through thickness) extending over part of the beam width located at the centre (Figure. [1]). It is assumed that, wherever there is a crack, the beam is segmented, but connected by the spring.

From Castigliano’s theorem, the finite change in bending rotation, \( \theta \), due to the presence of crack is given by

\[
\theta = \frac{\partial u}{\partial M}
\]  
(1)

where, \( u \) is the strain energy because of the presence of crack and \( M \) is the applied bending moment.

The flexibility, \( C \), because of the crack is given as

\[
C = \frac{1}{k_r} = \frac{\partial \theta}{\partial M} = \frac{\partial^2 u}{\partial M^2}
\]  
(2)

Integrating Equation. [2],

\[
u = \frac{M^2}{2k_r}
\]  
(3)

The relation between the strain energy/unit length, \( \psi \), and bending moment can also be expressed as

\[
\psi = \frac{M^2}{2EI} \Rightarrow M^2 = 2EI\psi
\]  
(4)
Substituting for \( M \) in Equation. [3]

\[
\text{Strain energy, } u = \frac{E I \psi}{k_y}
\]  

(5)

Using perturbation theory, Gudmundson [8] derived the relation between the eigen values representing frequencies and the strain energies of the uncracked and cracked structure as

\[
\frac{\omega_n^2}{\omega_n^{'2}} = 1 - \frac{u_n}{u_0n}
\]  

(6)

where,

\( \omega_n \) = nth mode natural frequency of the cracked structure,
\( \omega_n' \) = nth mode natural frequency of the uncracked structure,
\( u_n \) = increase nth mode strain energy due to the finite bending at the crack, equal to the strain energy stored in the spring, given by Equation.[ 5], and,
\( u_0n \) = is the strain energy of the undamaged structure in nth mode.


\[
\frac{\Delta \omega_n}{\omega_n} = \frac{u_n}{2u_0n}
\]  

(7)

The strain energy of the uncracked structure, \( u_0n \), in case of beam structures is given by

\[
u_0n = \int_0^1 \psi(\beta) d\beta
\]  

(8)


\[
\frac{\Delta \omega_n}{\omega_n} = \frac{g_n(\beta)}{2K}
\]  

(9)

where,

\[
g_n(\beta) = \int_0^1 \frac{\psi_n(\beta)}{\psi_n(\beta)} d\beta
\]  

(10)

and, \( K = \frac{k_r L}{E I} \)  

(11)

If \( \Phi_n \) is the mode shape of the beam in its nth mode, the equation for \( \psi_n \) can be obtained as

\[
\psi_n(\beta) = \frac{1}{2} E I \Phi_n^{'2}(\beta)
\]  

(12)

For a beam with multiple cracks, according to linear superposition principle, Equation. [9] can be written as

\[
\left\{ \frac{\Delta \omega_n}{\omega_n} \right\}_{n \times 1} = \left\{ H \right\}_{n \times m} \left\{ S \right\}_{m \times 1}
\]  

(13)

where, \( n \) is number of modal frequencies used, \( m \) is number of beam segments, \( \{ S \} \) is the damage parameter proportional to the flexibility of each segment, \( S_m \) and \( H = \sum_{i=1}^n g_n(\beta_i) \)

Here, [6, 7] used the mode shape equations obtained from basic vibration theory, to compute the values of specific strain energy \( \psi_n(\beta) \) in Equation. [12], where the former obtains them using symbolic computation and the latter derives them manually. In the present approach, the curvature terms on the right hand side of Equation. [10] are obtained by numerical
differentiation of "measured" mode shapes, whose values, in this case, are provided by the FEA. The damage parameter vector \([S]\) is calculated from Equation. [13] using pseudo-inverse technique and \(K\) Vs \(\beta\) is plotted for every mode. This process is repeated for all segments with non-zero \(S\) value. Since the location of the crack and the value of the spring stiffness, which is a function of crack size, have to be unique, all the \(K\) vs \(\beta\) curves should pass through a common point on the \(K\), \(\beta\) space. Thus the location of the crack and spring stiffness \(K\) are determined by the point of intersection of the \(K\) vs \(\beta\) curves of the different modes.

**DETERMINATION OF CRACK LENGTH**

The relation between the bending spring constant, \(k\), and extent of crack in case of beams containing through width partial thickness cracks \((a/h)\) has been derived earlier [3, 9]. Here a similar methodology is applied to determine the crack size from the spring stiffness by equating the spring energy to the energy stored in the crack, which is related to the Stress Intensity Factor (SIF) at the crack tip.

If \(K_i\) is the SIF, \(E\) the Young’s modulus and \(A\) the area of the crack, then, the change in elastic deformation energy due to the crack, as given by [10], is

\[
\Delta U = \frac{1}{E} \int K_i^2 dA
\]

The SIF equations including the finite width correction factor, \(f(\gamma)\), for different width to thickness ratios derived by Boduroglu and Erdogan [11] is expressed as

\[
K_i = \frac{6M_0}{t^2} f(\gamma) \sqrt{b} = \frac{3M}{w t^2} f(\gamma) \sqrt{b}
\]

where, \(M_0\) is the bending moment/unit width, \(t\) is the thickness of the beam and \(f(\gamma)_{w/t=7.8} = 25.77\gamma^2 + 6.81 \gamma^3 - 32.83 \gamma^4 + 18.3 \gamma^5 + 3.83 \gamma + 1.01\).

After incorporating Equation. 15 in Equation. 14

\[
\Delta U = \frac{9M}{E w^2 t^2} \int_0^b f(\gamma)^2 b^2 dA
\]

The change in strain energy can also be defined as

\[
\Delta U = \frac{1}{2} M \theta = \frac{1}{2} \frac{M^2}{k_r}
\]

Comparing Equations. 16 and 17

\[
Spring\ constant, k_r = \frac{E t^3}{18 \gamma^2 g(\gamma)}
\]

\[
g(\gamma)_{w/t=7.8} = 110.7\gamma^{10} + 63.8\gamma^9 - 329.1\gamma^8 + 110.2\gamma^7 + 282.4\gamma^6 - 343.3\gamma^5 + 200\gamma^4 - 82.5\gamma^3 + 25.8\gamma^2 - 5.1\gamma + 1
\]

In Equations. (15,16,17,18), \(=b/w\), where \(b\) is the semi-crack length and \(w\) is the half width of the beam. Equation. 18 can be solved to obtain \(b/w\) from known values of \(k_r\).

**NUMERICAL STUDY**

The above explained theory is applied to frequency values of cantilever beams containing two cracks. The frequency values were obtained from commercial Finite Element Analysis (FEA) software, ANSYS 10. For this numerical study, a beam 600mm long, 50mm wide and 3.2mm thick is considered. It is assumed to be made of Aluminium, Al-7076, with Young’s modulus 69.6GPa and density 2.77×10⁻⁶kg/mm³. Cracks of different sizes are modelled in two different locations on the beam. The first 6 natural frequency values, shown in Table. 1, are input to the damage detection algorithm. All computations including calculation of \(g(\beta)\) are carried out using MATLAB.

In this damage detection technique, the beam is divided into \(m\) segments. Referring to Equation. 13, using the percentage changes in each frequency values due to the presence of crack, the damage parameter vector \((S)\) is calculated. Now, the
percentage change in frequency is recalculated based on each of the obtained damage parameter, $S_m$. In our case, since the beam contains two cracks, only two of the m segments contain a non-zero damage parameter and thus two sets of percentage change in frequency values are obtained. Using this new percentage change in frequency values and Equation. [9], K Vs $\beta$ curves are obtained for each of the segments identified as containing damage.

Table 1: Natural frequencies (in Hz.) of undamaged and damaged beam

<table>
<thead>
<tr>
<th>Test case</th>
<th>Actual data</th>
<th>Predicted data</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>$\gamma_1$</td>
<td>$\beta_2$</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0.05</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>0.2</td>
<td>0.65</td>
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<td>0.4</td>
<td>0.55</td>
</tr>
<tr>
<td>8</td>
<td>0.15</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>9</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

In this study, the beam is divided into 10 segments and the above mentioned procedure is followed to obtain K Vs $\beta$ curves. Irrespective of the mode of vibration used, the spring which was modelled to represent the crack must be of the same stiffness. Hence, the location and spring stiffness is deduced from the point of intersection of K Vs $\beta$ curves of all modes. From spring stiffness, the crack length is the calculated using Equation. [18]. A comparison of the actual and predicted location as well as b/w ratio is shown in Table 2. Here, error is calculated as the difference between the predicted and actual value expressed as a percentage of the beam length and beam width when calculating crack location error and crack size error respectively. This helps reduce the multiplicity when comparing with very small actual values [12]. Typical K Vs $\beta$ curves are also shown in Figure. 2.

Table 2: Comparison of actual and predicted crack locations and crack sizes
Conclusion

A method to predict the location and size of cracks in cantilever beam using energy formulation has been presented. This method employs measured natural frequencies of the structure before and after damage and the mode shapes of the undamaged structure which can be obtained from measurements or FEA. Thus, this method can be easily adopted for damage detection in real complex structures. The accuracy of the method in characterising the damage is demonstrated with results obtained from numerical simulation using FEA. The maximum error in prediction of location and crack length is 5%. The method is currently being extended to detection of damage in plate structures.

References